Advanced Calculus I: Workshop 4

Exercise 1

Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of real numbers defined recursively by:

- $a_0 = 0$.
- $\forall n \in \mathbb{N},\ a_{n+1} = a_n + (-1)^n n^2$.

(1) Express with quantifiers what it means for (a_n) *not* to be a Cauchy sequence.
(2) Show that (a_n) is not a Cauchy sequence.
(3) Is (a_n) a convergent sequence?

Exercise 2

The purpose of this exercise is to prove the following fundamental statement:

Any bounded sequence of real numbers has a convergent subsequence.

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers, and consider the set $A = \{a_n,\ n \in \mathbb{N}\}$.

(1) In this question, we assume that the set A contains a finite number of elements. Show that (a_n) has a convergent subsequence.
(2) In this question, we assume that the set A is infinite. Show that, in this case also, (a_n) has a convergent subsequence.
[Hint: use the Bolzano-Weierstrass theorem to prove that A has an accumulation point $a \in \mathbb{R}$, then construct a subsequence of (a_n) that converges to a by exploiting the definition of an accumulation point.]