Advanced Calculus I: Workshop 3

Exercise 1 *(A characterization of the supremum using limits).*

Let A be a subset of $\mathbb{R}$ which is non empty and bounded from above.

(1) Why does A admit a supremum?

(2) Show that, for any $x \in \mathbb{R}$, $x = \sup(A)$ if and only if the following two properties hold:
- x is an upper bound for A,
- for any $\varepsilon > 0$, there exists $a \in A$ such that $x - \varepsilon < a \leq x$.

(3) By using the previous question, show that, for any $x \in \mathbb{R}$, $x = \sup(A)$ if and only if the following two properties hold:
- x is an upper bound for A,
- there exists a sequence (a_n) of elements of A such that $a_n \to x$.

Exercise 2 *(The 'Sandwich Theorem')*

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be three sequences of real numbers satisfying the following inequality:

$$\forall n \in \mathbb{N},\ \ b_n \leq a_n \leq c_n.$$

Show that, if (b_n) and (c_n) converge to the same limit $\ell \in \mathbb{R}$, then (a_n) also converges to ℓ.