Advanced Calculus I: Workshop 2

Exercise 1:

In this exercise, we are interested in the set of *algebraic numbers*. An algebraic number is a real number $x \in \mathbb{R}$ which arises as a root of a polynomial with integer coefficients i.e.such that there exist $k \in \mathbb{N}^*$ and $a_0, ..., a_k \in \mathbb{Z}$ with:

$$a_k x^k + a_{k-1} x^{k-1} + ... + a_1 x + a_0 = 0.$$

For instance, $\sqrt{2}$ is an algebraic number, since it is a root of the polynomial $X^2 - 2$. On the other hand, one can show (but it is difficult) that π, e, and $\log(2)$ are not algebraic (they are called *transcendental* numbers).

(1) Show the following variant of a proposition of the lectures: let S be a countable set, and let $\{A_s\}_{s\in S}$ be a family of countable sets indexed by S. Then the union set $\bigcup_{s\in S} A_s$ is countable.

(2) Show that the set $\mathcal{P}_k := \left\{a_k X^k + a_{k-1} X^{k-1} + ... + a_1 X + a_0,\ \ a_0, ..., a_k \in \mathbb{Z}\right\}$ of polynomials of order k with integer coefficients is countable.

(3) Show that the set of algebraic numbers is countable.

[Hint: remark that, for a given polynomial with integer coefficients P, the set $\{x \in \mathbb{R},\ P(x) = 0\}$ is countable, then use Questions (1) and (2).]

Exercise 2: Let A and B be two bounded subsets of $\mathbb{R}$.

(1) Let $-A$ be the subset of $\mathbb{R}$ defined by:

$$-A = \{-a,\ a \in A\}.$$

Show that $\sup(-A) = -\inf A$.

(2) Let $A + B$ be the subset of $\mathbb{R}$ defined by:

$$A + B = \{a + b,\ a \in A,\ b \in B\}.$$

Show that $\sup(A + B) = \sup A + \sup B$.

Let now X be a non empty subset of $\mathbb{R}$, and let $f, g : X \to \mathbb{R}$ be two functions with bounded range.

(3) Show that:

$$\sup\{f(x) + g(x),\ x \in X\} \leq \sup\{f(x),\ x \in X\} + \sup\{g(x),\ x \in X\}.$$

(4) Construct an example where strict inequality holds.