Advanced Calculus I: Workshop 1

Exercise 1: Let A, B and C be three sets, and $f : A \to B$, $g : B \to C$ two functions.

(1) Show that, if the composed function $g \circ f$ is one-to-one, then f is one-to-one.
(2) Is the converse of this property true? That is, is it true that, if f is one-to-one, then $g \circ f$ also is? If your answer is yes, prove it; else, provide a counter-example.
(3) Show that, if the composed function $g \circ f$ is onto, then g is onto.
(4) Does the converse of this property hold? If your answer is yes, prove it; else, provide a counter-example.

Exercise 2: Let $\{a_i\}_{i=1,\ldots,n}$ and $\{b_i\}_{i=1,\ldots,n}$ be two sequences of real numbers. Prove the *Cauchy-Schwarz inequality* by induction on n:

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$