Advanced Calculus I: Revisions for the final exam

Exercise 1:
Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Prove that there exist two real numbers $m, p \in \mathbb{R}$ such that:

$$\forall x \in \mathbb{R}, \ \ |f(x)| \leq m|x| + p.$$

Exercise 2:
Let $f : (0,1) \to \mathbb{R}$ be an increasing function, which is bounded from above. Show that the limit $\lim\limits_{x\to 1} f(x)$ exists.

Exercise 3:
Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Show that there exists $c \in \left[0, \frac{1}{2}\right]$ such that:

$$f(c) = f\left(c + \frac{1}{2}\right).$$

Exercise 4:

(1) Give an example of a continuous and bounded function $f : (0,1) \to \mathbb{R}$ which has neither a maximum, nor a minimum on $(0,1)$.
(2) Give an example of a bounded function $f : [0,1] \to \mathbb{R}$ which has neither a maximum, nor a minimum on $[0,1]$.

Exercise 5:
Let $f : [0,1] \to \mathbb{R}$ be a differentiable function, whose derivative is continuous, such that $f(0) = 0$, and, for all $x \in [0,1]$, $f'(x) > 0$. Show that there exists a real number $m > 0$ such that:

$$\forall x \in [0,1], \ \ f(x) \geq mx.$$

Exercise 6: *(Around the constant of Euler-Mascheroni)*

(1) By using the mean-value theorem, show that, for any natural number $n \in \mathbb{N}^*$, one has:

$$\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}.$$

Let $\{x_n\}_{n\in\mathbb{N}^*}$ be the sequence defined by:

$$x_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \log(n).$$

(2) Show that the sequence $\{x_n\}_{n\in\mathbb{N}^*}$ is strictly decreasing.
(3) Show that, for any $n \in \mathbb{N}^*$, one has:

$$0 \leq x_n \leq 1.$$

(4) Conclude that $\{x_n\}_{n\in\mathbb{N}^*}$ has a limit $\gamma \in [0,1)$. This limit is called the *Euler-Mascheroni constant.*

Exercise 7:
Let $a < b$ be two real numbers, and let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f(a) = f(b)$ and $f'(a) = f'(b) = 0$. By applying Rolle's theorem to the auxiliary function $h(x) = e^{-x}(f(x) + f'(x))$, show that there exists a number $c \in (a,b)$ such that:

$$f''(c) = f(c).$$

Exercise 8:
Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions. We assume that:

$$\forall x \in \mathbb{Q}, \ \ f(x) < g(x).$$

(1) Show that, for all $x \in \mathbb{R}$, $f(x) \leq g(x)$.
[Hint: Start by observing that, for any real number x, there exists a sequence $\{r_n\}$ of elements of $\mathbb{Q}$ such that $r_n \to x$.]
(2) Does it necessarily hold that, for all $x \in \mathbb{R}$, one has: $f(x) < g(x)$? If your answer is yes, prove it; else, provide a counterexample.

Exercise 9:
Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$\forall x \in \mathbb{R}, \ f(x) = \frac{1}{3}(4 - x^2).$$

Let also $\{u_n\}_{n\in\mathbb{N}}$ be the sequence defined recursively by:
- $u_0 = \frac{1}{2}$,
- $\forall n \in \mathbb{N}, \ u_{n+1} = f(u_n)$.

(1) Calculate u_1, u_2.
(2) Show by induction that, for any $n \in \mathbb{N}$, $u_n \in \left[0, \frac{4}{3}\right]$.
(3) Calculate the derivative of f.
(4) Show that, for any $x \in \left[0, \frac{4}{3}\right]$, one has:

$$|f(x) - 1| \leq \frac{8}{9}|x - 1|.$$

[Hint: apply the mean-value theorem to f.]
(5) Infer from your answer to the previous question that, for any $n \in \mathbb{N}$:

$$|u_{n+1} - 1| \leq \frac{8}{9}|u_n - 1|.$$

(6) Show that, for any $n \in \mathbb{N}$, $|u_n - 1| \leq \left(\frac{8}{9}\right)^n |u_0 - 1|$.
(7) Conclude that $\{u_n\}_{n\in\mathbb{N}}$ converges to 1.

Exercise 10:
For any natural number $n \geq 2$, let $f_n : [1, +\infty) \to \mathbb{R}$ be the function defined by:

$$f_n(x) = x^n - x - 1.$$

(1) Show that, for a given $n \geq 2$, the function f_n is strictly increasing on $[1, +\infty)$.
(2) Show that, for a given $n \geq 2$, there exists a unique real $x_n \in [1, +\infty)$ such that $f_n(x_n) = 0$.
(3) Show that, for any $n \geq 2$, one has: $f_{n+1}(x_n) > 0$.
(4) Infer that the sequence $\{x_n\}$ is decreasing.
(5) Show that the sequence $\{x_n\}$ has a limit ℓ.
(6) Show that $\ell = 1$.
[Hint: Argue by contradiction; if $\ell \neq 1$, show that there is a fixed number $\alpha > 0$ and a rank $N \in \mathbb{N}$ in the sequence such that, for $n \geq N$, $x_n > 1 + \alpha$, and infer a contradiction from this last fact.]

Exercise 11:
Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $f(1) > 0$.
(1) Let $C \subset [0, 1]$ be defined by:

$$C = \{x \in [0, 1], \ f(x) = 0\}.$$

Show that C is compact.
(2) infer that there exists a number $x_0 \in [0, 1)$ such that:

$$f(x_0) = 0 \text{ and } \forall x > x_0, \ f(x) > 0.$$

Exercise 12:
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the following property:

$$\forall x, y \in \mathbb{R}, \ f(x + y) = f(x) + f(y).$$

Denote as $m = f(1)$.

(1) Show that, for any rational number $x \in \mathbb{Q}$, one has $f(n) = mx$.
[Hint: Start by proving this property for $x \in \mathbb{N}$, then for $x \in \mathbb{Z}$.]
(2) infer that, for any real number $x \in \mathbb{R}$, one has $f(x) = mx$.

Exercise 13**:**

Let $a < b$ be two real numbers, and $f, g : [a, b] \to \mathbb{R}$ be two continuous functions such that:

$$f(a) \leq g(a) \text{ and } f(b) \geq g(b).$$

Show that the equation $f(x) = g(x)$ has a solution in $[a, b]$.

Exercise 14

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *even* if, for any $x \in \mathbb{R}$, $f(x) = f(-x)$, and is said to be *odd* if $f(x) = -f(-x)$.

(1) Give examples of non constant even and odd functions, and draw their graphs.
(2) Show that the derivative f' of a differentiable, odd function, is even.
(3) Does the converse necessarily hold (i.e. if f' is even, is f necessarily odd)? If your answer is yes, prove it; else, provide a counterexample.
(4) Show that the derivative f' of a differentiable, odd function, is even.