Advanced Calculus I: Revisions for Midterm 2; solution of Exercise 7

Exercise 7:

The purpose of this exercise is to provide an alternative proof of the Heine theorem. Let $K \subset \mathbb{R}$ be a compact set, and let $f : K \to \mathbb{R}$ be a continuous function. The proof goes by a contradiction argument.

(1) Negate the definition of uniform continuity for f.

(2) Show that, if f is not uniformly continuous on K, then there exist $\varepsilon > 0$, as well as two sequences $\{x_n\}_{n\in\mathbb{N}^*}$ and $\{y_n\}_{n\in\mathbb{N}^*}$ of elements of K which satisfy the following properties:
$$\forall n \in \mathbb{N}^*,\ \ |x_n - y_n| < \frac{1}{n},\ \text{and } |f(x_n) - f(y_n)| > \varepsilon.$$

(3) Show that there exist two subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}_{n\in\mathbb{N}^*}$ and $\{y_n\}_{n\in\mathbb{N}^*}$ respectively, which converge to a common limit $\alpha \in K$.

(4) End the proof by obtaining a contradiction between this fact and the properties of Question (2).

Solution:

(1) Saying that f is uniformly continuous over K reads, in terms of quantifiers:
$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \ \forall x, y \in K,\ \ |x - y| < \delta \Rightarrow |f(x) - f(y| < \varepsilon.$$
Hence, saying that f is not uniformly continuous over K can be written as:
$$\exists \varepsilon > 0,\ \forall \delta > 0,\ \ \exists x, y \in K,\ \text{such that } |x - y| < \delta \text{ and } |f(x) - f(y| > \varepsilon,$$
that is: 'there is (at least) one value of ε for which there exists a pair (x, y) of elements of K that are arbitrarily close from one another, but whose images $f(x), f(y)$ are distant from at least ε.'

(2) We just explained why there exists $\varepsilon > 0$ such that:
$$\forall \delta > 0,\ \ \exists x, y \in K,\ \text{such that } |x - y| < \delta \text{ and } \ \Rightarrow |f(x) - f(y| > \varepsilon.$$
In particular, taking $\delta = \frac{1}{n}$, for $n = 1, ...,$ we obtain that, for any $n \in \mathbb{N}^*$, there exist $x_n, y_n \in K$ such that:
$$|x_n - y_n| < \frac{1}{n},\ \text{and } |f(x_n) - f(y_n)| > \varepsilon.$$

(3) $\{x_n\}$ is a sequence of elements of the compact set K, so it has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}^*}$ which converges to an element $\alpha \in K$. Then, the corresponding subsequence $\{y_{n_k}\}_{k\in\mathbb{N}^*}$ of $\{y_n\}$ also converges to α, since we have:
$$\begin{array}{rcl} \forall k \in \mathbb{N}^*,\ \ |y_{n_k} - \alpha| & = & |y_{n_k} - x_{n_k} + x_{n_k} - \alpha| \\ & \leq & |y_{n_k} - x_{n_k}| + |x_{n_k} - \alpha| \\ & \leq & \frac{1}{n_k} + |x_{n_k} - \alpha| \end{array},$$
and the first term at the right-hand side obviously converges to 0, while the second one also does, by definition of α and the subsequence $\{x_{n_k}\}$. This proves the desired result.

(4) With the help of the previous questions, we are in position to obtain a contradiction with the continuity of f. Indeed, we know that, since f is continuous over K, for any sequence $\{z_n\}$ of elements of K which converges to α, the sequence $\{f(z_n)\}$ converges to $f(\alpha)$. But, using the two sequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of the previous question, which both converge to α leads to a contradiction, for we know, by construction of these sequences that:
$$\forall k \in \mathbb{N}^*, |f(x_{n_k}) - f(y_{n_k})| > \varepsilon.$$
Passing to the limit in this inequality, we obtain:
$$0 = |f(\alpha) - f(\alpha)| \geq \varepsilon,$$
which is impossible, since $\varepsilon > 0$. Contradiction, and f is uniformly continuous.