Advanced Calculus I: revision exercises for Midterm 1.

Exercise 1

For each of the following statements, state whether it is true or false; if it is true, prove it (or invoke a theorem of the lectures), and if it is false, provide a counterexample.

(1) Each sequence $\{a_n\}$ of real numbers has a convergent subsequence.
(2) If a sequence $\{a_n\}$ of real numbers is increasing and not bounded from above, then it goes to $+\infty$.
(3) A sequence $\{a_n\}$ of real numbers is convergent if and only if it is bounded.
(4) A sequence $\{a_n\}$ of real numbers is convergent if and only if it is Cauchy.

Exercise 2

Let $\{a_n\}$ be a sequence of integers; show that, if $\{a_n\}$ converges, then is is stationary after a certain rank.

Exercise 3

Show that, if A is uncountable, and B is countable, then $A \cap B$ is countable.

Exercise 4

Let $\{a_n\}$ be a monotone sequence of real numbers. Show that if $\{a_n\}$ has a convergent subsequence, then it is convergent. Is it true in the case that $\{a_n\}$ is not monotone?

Exercise 5

Let $A \subset \mathbb{R}$ be a non empty set which is bounded from above.

(1) Does A have a maximum? A supremum?
(2) Show that, if $x \in A$ is such that $x < \sup(A)$, then $\sup(A \setminus \{x\}) = \sup(A)$.
(3) Show that, if $x \in A$ is such that $\sup(A \setminus \{x\}) < \sup(A)$, then $x = \sup(A)$.

Exercise 6

(1) Recall the definition of a Cauchy sequence.
(2) Show the identity:

$$\forall x, y \in \mathbb{R}, \ \sin(x) - \sin(y) = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right).$$

[Hint: introduce $p = \frac{x+y}{2}$, $q = \frac{x-y}{2}$, and remark that $x = p + q$, $y = p - q$.]

(3) Deduce from the answer to (2) that:

$$\forall x, y \in \mathbb{R}, \ |\sin(x) - \sin(y)| \leq |x - y|.$$

(4) Let $b \in (0, 1)$ and $c \in \mathbb{R}$. Let $\{a_n\}_{n \in \mathbb{N}}$ be the sequence of real numbers defined by:

$$u_0 \in \mathbb{R}, \text{ and } \forall n \in \mathbb{N}, \ a_{n+1} = b \sin(a_n) + c.$$

Show that, for all $n \in \mathbb{N}$,

$$|a_{n+1} - a_n| \leq b^n |a_1 - a_0|.$$

(5) Show that $\{a_n\}$ is a Cauchy sequence.
[Hint: For $n < m$, decompose $|a_m - a_n| \leq |a_m - a_{m+1}| + ... + |a_{n+1} - a_n|$, and use the previous question with the inequality $1 + b + b^2 + ... + b^n \leq \frac{1}{1-b}$.]

(6) Infer that $\{a_n\}$ converges.