Advanced Calculus I: revision exercises for Midterm 1.

Exercise 1 Let A and B be two non empty bounded subsets of $\mathbb{R}$.

(1) Let $-A$ be the subset of $\mathbb{R}$ defined by:

$$-A = \{-a,\ a \in A\}\,.$$

Show that $\sup(-A) = -\inf A$.

(2) Let $A + B$ be the subset of $\mathbb{R}$ defined by:

$$A + B = \{a + b,\ a \in A,\ b \in B\}\,.$$

Show that $\sup(A + B) = \sup A + \sup B$.

Let now X be a non empty subset of $\mathbb{R}$, and let $f, g : X \to \mathbb{R}$ be two functions with bounded range.

(3) Show that:

$$\sup\{f(x) + g(x),\ x \in X\} \leq \sup\{f(x),\ x \in X\} + \sup\{g(x),\ x \in X\}\,.$$

(4) Construct an example where strict inequality holds.

Exercise 2 Let A and B be two non empty bounded subsets of $\mathbb{R}$ such that $A \subset B$.
Compare the four numbers $\inf(A)$, $\sup(A)$, $\inf(B)$ and $\sup(B)$.

Exercise 3 Let A be a non empty subset of $\mathbb{R}$ which is bounded from below.

(1) Why does the infimum $\inf(A)$ exist?

(2) Let B be the subset of $\mathbb{R}$ defined by:

$$B = A \cap (-\infty, \inf(A) + 1).$$

Show that B is non empty, bounded from below, and that $\inf(B) = \inf(A)$.

Exercise 4 Let A, B, C, D be four sets, and let $f : A \to C$, $g : B \to D$ be two functions. Define the function $h : A \times B \to C \times D$ by:

$$\forall (a, b) \in A \times B,\ \ h(a, b) = (f(a), g(b)).$$

(1) Show that, if f and g are one-to-one, then so is h.

(2) Show that, if f and g are onto, then so is h.

(3) Show that, if f and g are bijective, then so is h.

Exercise 5 Let A and B be two sets such that there exists a surjective function $f : A \to B$.

(1) Show that, if B is infinite, then so is A.

(2) Show that, if B is uncountable, then so is A.

Exercise 6 *(This exercise is more difficult and should be adressed after the basic exercises over countability have been mastered).* If A is an arbitrary set, one defines the *power set* $\mathcal{P}(A)$ as the set of all the subsets of A. For instance, if $A = \{0, 1, 2, \}$, then:

$$\mathcal{P}(A) = \{\,\{0\}\,,\{1\}\,,\{2\}\,,\{0,1\}\,,\{0,2\}\,,\{1,2\}\,,\{0,1,2\}\}\,.$$

(1) Show that, if A is a countably infinite set, then $\mathcal{P}(A)$ is uncountable.

(2) Show that, if two sets A and B are equivalent, then so are $\mathcal{P}(A)$ and $\mathcal{P}(B)$.

Exercise 7 Let $(a_n)_{n\in\mathbb{N}}$ be the sequence recursively defined by:

$$a_0 = 3 \text{ and, for } n \geq 0,\ \ a_{n+1} = \frac{1}{4 - a_n}.$$

(1) Show that, for any $n \in \mathbb{N}$, $0 < a_n < 4$.

(2) Show that (a_n) is a decreasing sequence.
(3) Show that (a_n) converges, and find its limit.

Exercise 8 Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of real numbers defined by:

$$\forall n \in \mathbb{N},\ \ a_n = (-1)^n.$$

Show that (a_n) is divergent.

Exercise 9 Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of real numbers defined by:

$$\forall n \in \mathbb{N},\ \ a_n = n + (-1)^n n.$$

(1) Show that (a_n) is divergent.
(2) Find a convergent subsequence to (a_n).

Exercise 10 *(This exercise is more difficult and should be adressed after the basic exercises over sequences have been mastered).* Let $(a_n)_{n\in\mathbb{N}^*}$ be a sequence of real numbers; the *Césaro* mean associated to (a_n) is the sequence $(b_n)_{n\in\mathbb{N}^*}$, defined by:

$$\forall n \in \mathbb{N},\ \ b_n = \frac{a_1 + ...a_{n-1} + a_n}{n}.$$

(1) Show that if (a_n) converges to a real number ℓ, then so does (b_n).
(2) Find an example of a sequence (a_n) which is divergent, but such that (b_n) converges.

Exercise 11 Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers; for each of the following statements, say if it is true or false. If its is true, prove it; else, exhibit a counterexample.

(1) If (a_n) is increasing, then $a_n \to +\infty$.
(2) If $a_n \to -\infty$, then $|a_n| \to +\infty$.
(3) If (a_n) is increasing, then it has a convergent subsequence.
(4) If (a_n) is not bounded, then it does not have a convergent subsequence.

Exercise 12 Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of real numbers, such that (a_n) is bounded and $b_n \to 0$. Show that $a_n b_n \to 0$.