# Advanced Calculus I: Homework 6

Assigned 10/16/2014, due 10/23/2014.

**Exercise 1**

Let $D$ be a subset of $\mathbb{R}$, $f : D \to \mathbb{R}$ be a function, and $x_0$ be an accumulation point of $D$. Draw an example of a situation corresponding to each of the following statements, and express what it means in terms of quantifiers:

(1) $f$ has a limit $\ell \in \mathbb{R}$ at $x_0$.
(2) $f$ has a left-limit $\ell \in \mathbb{R}$ at $x_0$.
(3) $f$ has a right-limit $\ell \in \mathbb{R}$ at $x_0$.
(4) $f$ goes to $+\infty$ when $x \to x_0$.
(5) $D = (0, +\infty)$ and $f$ has a finite limit $\ell \in \mathbb{R}$ when $x$ goes to $+\infty$.

**Exercise 2**

Let $f : (0, 1) \to \mathbb{R}$ be the function defined by $f(x) = \frac{\sqrt{9-x}-3}{x}$. Show that $f$ has a limit at $x_0 = 0$, and calculate this limit.

**Exercise 3**

Calculate the following limits:

(1) $\lim_{x\to 0} \frac{x^3-2x+1}{x-1}$ (2) $\lim_{x\to +\infty} \frac{x^3-2x+1}{x-1}$
(3) $\lim_{x\to 1} \frac{x^3-2x+1}{x-1}$ (4) $\lim_{x\to 0} \frac{x^3-2x^2+x}{x^2-x}$
(5) $\lim_{x\to 0} \frac{x^3-2x+1}{x^2-x}$

*[Hint: For (3), notice that $x^3 - 2x + 1 = (x - 1)(x^2 + x - 1)$.]*

**Exercise 4**

Let $D$ be a subset of $\mathbb{R}$, $f : D \to \mathbb{R}$ be a function such that, for any $x \in D$, $f(x) \neq 0$, and $x_0$ be an accumulation point of $D$. We assume that $f$ has a limit $\ell \neq 0$ at $x_0$. The purpose of this exercise is to show by two different methods the following property, called $(*)$:

$$(*) \quad (\exists \delta > 0), (\exists M > 0), \forall x \in D \cap (x_0 - \delta, x_0 + \delta), \; |f(x)| \geq M.$$

(1) Explain what $(*)$ means using a drawing.
(2) *First method:* Show this property by using directly the definition of the limit of a function.
(3) *Second method: using sequences*
   (a) Express the *negation* of the property $(*)$ in terms of quantifiers.
   (b) We now argue by contradiction, assuming that $(*)$ does not hold. Use the negation of the previous question to construct a sequence $(x_n)$ of points of $D$ such that, for all $n \in \mathbb{N}$, $x_n \neq x_0$ and $x_n \to x_0$,
   $$|f(x_n)| \leq \frac{1}{n}.$$
   (c) Conclude.

**Exercise 5** *(This exercise is reprinted from [Gaughan], ex. 22 p. 80)*

Show an example of functions $f$ and $g$ which fail to have limits at a point $x_0$, but such that $f + g$ has a limit at $x_0$. Give similar examples for $fg$ and $\frac{f}{g}$.

**Exercise 6**

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 < x \leq 1, \\ 2 - (x-1)^2 & \text{if } x > 1. \end{cases}$$

(1) Draw the function $f$.

(2) Calculate the left- and right-limits $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x)$ and $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x)$ at $x_0 = 0$ and $x_0 = 1$.

(3) At which points $x_0 \in \mathbb{R}$ is $f$ continuous?