Advanced Calculus I: Homework 5

Assigned 10/09/2014, due 10/16/2014.

In the exercises ahead, unless otherwise specified, you are authorized to use the contents of Lecture XII, around operations over limits of functions. In particular, you do not need to resort to the ε-definition of limit to say that $\lim_{x\to x_0} x = x_0$, etc...

Exercise 1 Let $f:(0,1)\to\mathbb{R}$ be the function defined by $f(x)=x^2$, $g:(1,2)\to\mathbb{R}$ be given by $f(x)=3-x$, and $h:(0,2)\setminus\{1\}$ be defined by:

$$\forall x\in(0,2)\setminus\{1\}\,,\;\; h(x)=\begin{cases} f(x) & \text{if } x\in(0,1)\\ g(x) & \text{if } x\in(1,2)\end{cases}\;.$$

(1) Draw the three functions f,g,h.
(2) Show that f has a limit as $x\to 1$.
(3) Show that g has a limit as $x\to 1$.
(4) Show that h does not have a limit at 1.

Exercise 2

(1) Show that:

$$\lim_{x\to 1}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=1.$$

(2) Let $m,n\in\mathbb{N}^*$. Show that the function $f:(0,1)\to\mathbb{R}$ defined by:

$$f(x)=\frac{\sqrt{1+x^m}-\sqrt{1-x^m}}{x^n}$$

has limit 0 at $x=0$ if $m>n$, has limit 1 at $x=0$ if $m=n$, and has an infinite limit at $x=0$ if $m<n$.

(3) Calculate the limit:

$$\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}.$$

Exercise 3 *(Reprinted from Ex. 4 p. 79 in [Gaughan]).*

Give an example of a bounded function $f:\mathbb{R}\to\mathbb{R}$ which has a limit at every point but 2; use the definition of limit to justify your answer.

Exercise 4 *(Reprinted from Ex. 11 p. 79 in [Gaughan]: The 'Sandwich theorem', 'function' version).*

Let $D\subset\mathbb{R}$, $f,g,h:D\to\mathbb{R}$, and let x_0 be an accumulation point of D. Assume that the following inequality holds:

$$\forall x\in D,\;\; f(x)\le h(x)\le g(x),$$

and assume that f and g have limits at x_0 with:

$$\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x).$$

Show that h has a limit at x_0, and that:

$$\lim_{x\to x_0} h(x)=\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x).$$

Exercise 5

Let $f,g:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2+x-2$, and $g(x)=(x^2+x-2)\cos(x)$. The purpose of this exercise is to use the ε-definition of limit (and only the definition) to prove that f and g have limits at $x=1$ and calculate these limits..

(1) Show that, for any $0 < \varepsilon < 1$, and any $x \in \mathbb{R}$, if $|x-1|< \frac{\varepsilon}{4}$, then $|x^2+x-2|< \varepsilon$.
(2) Deduce from your answer to (1) that $\lim\limits_{x \to 1} f(x)$ and $\lim\limits_{x \to 1} g(x)$ exist and that they both equal 0.

$\boxed{\textbf{Exercise } 6}$ *(Partially reprinted from Ex. 12 p. 79 in [Gaughan]).*

(1) Show that, for any sequence $\{x_n\}$ of real numbers, if $x_n \to \ell$, then $|x_n| \to |\ell|$.
(2) Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ be a function, and let x_0 be an accumulation point of D. Infer from (1) that, if $\lim\limits_{x \to x_0} f(x)$ exists and equals $a \in \mathbb{R}$, then $\lim\limits_{x \to x_0} |f(x)|$ exists and equals $|a|$.