Advanced Calculus I: Homework 3

Assigned 09/25/2014, due 10/02/2014.

Exercise 1 *(Reprinted from Ex. 7 p. 55 in [Gaughan]).* Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Show that (a_n) converges to a real number a if and only if the sequence with general term $(a_n - a)$ converges to 0.

Exercise 2 *(Reprinted from Ex. 8 p. 55 in [Gaughan]).* Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $a_n \to a$, for some $a \in \mathbb{R}$, and define the new sequence $(b_n)_{n\in\mathbb{N}}$ by:

$$\forall n \in \mathbb{N},\ \ b_n = \frac{1}{2}(a_n + a_{n+1}).$$

Show that $b_n \to a$.

Exercise 3 *(Partially reprinted from Ex. 32 p. 56 in [Gaughan]).*
In each of the following cases, find the limit of the sequence (a_n):

(1) $a_n = \frac{n^2+2n}{n^2-8}$.
(2) $a_n = \frac{\cos(n)}{n}$.
(3) $a_n = \frac{\sin(n^2)}{\sqrt{n}}$.
(4) $a_n = \frac{n}{3n^2+2}$.
(5) $a_n = \left(\sqrt{4-\frac{1}{n}} - 2\right) n$.
(6) $a_n = (-1)^n \frac{\sqrt{n}}{n+7}$.
(7) $a_n = \sqrt{n^2+1} - n$.

Exercise 4 *(Reprinted from Ex. 25 p. 56 in [Gaughan]).* Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of real numbers. Assume that $a_n \to a$, where a is a real number different from 0, and that the product sequence $(a_n b_n)$ converges. Show that (b_n) is a convergent sequence.

Exercise 5 *(Reprinted from Ex. 10 p. 55 in [Gaughan]).* Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

(1) Show that, if (a_n) converges towards a real number a, then the sequence with general term $|a_n|$ converges to $|a|$.
(2) Is the converse property true? If your answer is yes, prove it; else, find a counterexample.

Exercise 6 *(Square roots and limits)*
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers (i.e. $a_n \geq 0$ for $n \in \mathbb{N}$) which converges to $a \in \mathbb{R}$.

(1) By using a Theorem of the lectures, justify that the limit a satisfies: $a \geq 0$.
(2) In this question, we assume that $a = 0$. By using the ε-definition of the limit, show that $\sqrt{a_n} \to 0$.
(3) In this question, we assume that $a > 0$. Show that:

$$\forall n \in \mathbb{N},\ \ |\sqrt{a_n} - \sqrt{a}| \leq \frac{1}{\sqrt{a}}|a_n - a|.$$

[Hint: you may consider using a trick already presented during the lectures.]

(4) Let now $(b_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $b_n^2 \to \ell$, for some real number ℓ. Is it true that (b_n) necessarily converges? If your answer is yes, prove it; else, show a counterexample.