Advanced Calculus I: Homework 2

Assigned 09/18/2014, due 09/25/2014.

Exercise 1 *(Around the absolute value...)*.

This exercise is meant as a revision over the notion of *absolute value*, which is the natural way to appraise the distance between two real numbers. Recall that the absolute value $|x|$ of a real number x is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

(1) Show that, for any real numbers x, y, $|xy| = |x||y|$.
(2) Show that, for any $x \in \mathbb{R}$, and $M > 0$, one has: $|x| \leq M$ if and only if $-M \leq x \leq M$.
(3) Show that, for any $x, y \in \mathbb{R}$ and any $\varepsilon > 0$, one has: $|x - y| \leq \varepsilon$ if and only if $y - \varepsilon \leq x \leq x + \varepsilon$.
(4) Show the *triangle inequality*:

$$\forall x, y \in \mathbb{R}, \ \ |x + y| \leq |x| + |y|.$$

(5) Show the *reverse triangle inequality*:

$$\forall x, y \in \mathbb{R}, \ \ ||x| - |y|| \leq |x + y|.$$

Exercise 2 Let A, B be two sets, with A countable, such that there exists a surjective function $f : A \to B$. Show that B is countable.

[Hint: use the fact that there exists a bijective function $g : \mathbb{N} \to A$ and construct an injective function $h : B \to \mathbb{N}$.]

Exercise 3 Let B be an uncountable set, and $A \subset B$ be a countable subset of B. Show that the complement $B \setminus A = \{b \in B \text{ s.t. } b \notin A\}$ is uncountable.

[Hint: argue by contradiction.]

Exercise 4 For each of the following subsets of $\mathbb{R}$, determine if it admits an infimum and / or a supremum, and provide it when it does:

(1) $[0, 1]$,
(2) $[0, 1)$,
(3) $\mathbb{Q}$,
(4) $\mathbb{Q} \cap [0, \sqrt{2})$,
(5) $\left\{\frac{1}{n}, \ n \in \mathbb{N}^*\right\}$,
(6) $\{(-1)^n, \ n \in \mathbb{N}\}$,
(7) $\left\{(-1)^n + \frac{(-1)^{n+1}}{n}, \ n \in \mathbb{N}^*\right\}$,
(8) $\left\{(-1)^n + \frac{1}{n}, \ n \in \mathbb{N}^*\right\}$,
(9) $\left\{\frac{1+x}{1+x^2}, \ x \in \mathbb{R}\right\}$.

Exercise 5

Let A and B be two subsets of $\mathbb{R}$ enjoying the following property:

$$\forall a \in A, \ \forall b \in B, \ \ a < b.$$

(1) Give an example of two such subsets of $\mathbb{R}$ where neither A nor B is the empty set.
(2) Show that A admits an upper bound, and B admits a lower bound.
(3) Show that $\sup A$ and $\inf B$ exist, and satisfy $\sup A \leq \inf B$.

(4) Give an example of two such subsets of $\mathbb{R}$ where neither A nor B is the empty set, where $\sup A = \inf B$. (i.e. the strict inequality is not preserved when passing to the sup / inf).

Exercise 6 *(Reprinted from Ex. 44 – 45 p. 29 in [Gaughan]).*
Let $S \subset \mathbb{R}$ be a set.

(1) If $x = \sup S$, show that, for any $\varepsilon > 0$, there exists an element $a \in S$ such that $x - \varepsilon < a \leq x$.
(2) If $y = \inf S$, show that, for any $\varepsilon > 0$, there exists an element $a \in S$ such that $y \leq a < y + \varepsilon$.