Advanced Calculus I: Homework 11

Assigned 11/20/2014, due 12/02/2014.

Exercise 1 *(Reprinted from Ex. 22 p. 131 in [Gaughan])*
By using the mean-value theorem, show that, for any $n \in \mathbb{N}^*$, for any real numbers $0 \leq y \leq x$, one has:

$$ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y)$$

Exercise 2 *(Reprinted from Ex. 20 p. 130 in [Gaughan])*
Let $f : [0,2] \to \mathbb{R}$ be a differentiable function such that f' is continuous and:

$$f(0) = 0, \;\; f(1) = 2, \;\text{ and } f(2) = 2.$$

Show that, successively:

(1) There exists $c_1 \in (0,2)$ such that $f'(c_1) = 0$.
(2) There exists $c_2 \in (0,2)$ such that $f'(c_2) = 2$.
(3) There exists $c_3 \in (0,2)$ such that $f'(c_3) = \frac{3}{2}$.

Exercise 3
Analyze the variations of the function $f : \mathbb{R} \to \mathbb{R}$ over $\mathbb{R}$ defined by $f(x) = x^5 - 5x + 1$, and prove that the equation $f(x) = 0$ has exactly 3 solutions on $\mathbb{R}$.

Exercise 4
Let $a > 0$ be a real number, and $f : [0,a] \to \mathbb{R}$ be a differentiable function such that:

$$f(0) = f(a) = f'(0) = 0.$$

Let also $g : [0,a] \to \mathbb{R}$ be defined as:

$$\forall x \in [0,a]\,, \;\; g(x) = \begin{cases} \frac{f(x)}{x} & \text{if } x \in (0,a]\,, \\ f'(0) & \text{if } x = 0 \end{cases} .$$

(1) Show that g is a continuous function on $[0,a]$.
(2) Show that g is differentiable on $(0,a)$ and calculate its derivative.
(3) State *precisely* Rolle's theorem.
(4) Show that there exists $c \in (0,a)$ such that $g'(c) = 0$, and infer from your answer that, for this particular value:

$$f(c) = cf'(c).$$

(5) Write the equation of the tangent line $y = mx + p$ to f at $x = c$.
(6) By using the results of Questions (4) and (5), prove that the tangent line to f at c passes through $(0,0)$.

Exercise 5
Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and let $x \in \mathbb{R}$ be a fixed real number. Show that there exists $c \in \mathbb{R}$ such that:

$$f(x) - f(-x) = x(f'(c) + f'(-c)).$$

[Hint: Introduce the auxiliary function $g(x) = f(x) - f(-x)$ and apply the Mean-value Theorem to g by noticing that $g(0) = 0$.]

Exercise 6
Let $a \in \mathbb{R}$, and let $f : [a,+\infty) \to \mathbb{R}$ be a continuous function, which is differentiable on $(a,+\infty)$. We also assume that $\lim_{x\to+\infty} f(x) = f(a)$. The purpose of this exercise is to prove that there exists a number $c \in (a,+\infty)$ such that $f'(c) = 0$.

(1) Let $g : [a, +\infty) \to \mathbb{R}$ be defined as $g(x) = f(x) - f(a)$. Show that g has a *maximum* and a *minimum* over $[a, +\infty)$, i.e. there exist $x_0, x_1 \in [a, +\infty)$ such that:

$$\forall x_0, x_1 \in [a, +\infty)\,,\ \ f(x_0) \leq f(x) \leq f(x_1).$$

(2) In the case that $f(x_0) = f(x_1) = 0$, conclude as for the desired result.

(3) When either $f(x_0)$ or $f(x_1)$ differs from 0, use the theorem about local extrema of functions to conclude.