Symmetric Determinantal Representations of Weakly-Skew Circuits

Bruno Grenet
ÉNS Lyon
U. of Toronto

Erich L. Kaltofen
North Carolina
State University

Pascal Koiran
ÉNS Lyon
U. of Toronto

Natacha Portier
ÉNS Lyon
U. of Toronto

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The problem

$$(x+y)+(y\times z) = \det \begin{vmatrix} 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ x & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{vmatrix}$$

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0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\\
x & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
y & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 1 & 0 & z & 0 & 0 & 0\\
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\end{vmatrix}$$

- Formal polynomial

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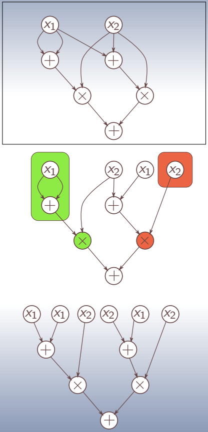
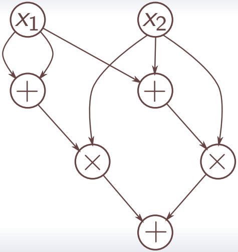
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{vmatrix}$$

- Formal polynomial
- Smallest possible dimension of the matrix

Representations of polynomials



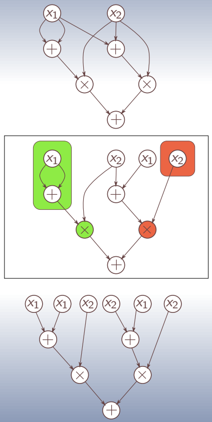
Arithmetic circuit:



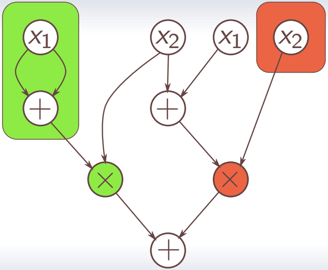
Size $e = 5$

Inputs $i = 2$

Representations of polynomials



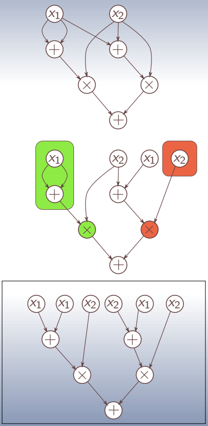
Weakly-skew circuit:



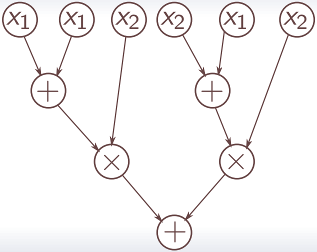
Size $e = 5$

Inputs $i = 4$

Representations of polynomials



Formula:



Size $e = 5$

Inputs $i = 6$

Motivation

L. G. Valiant, **Completeness classes in algebra**, STOC'79

Theorem (Universality of determinant and permanent)

Let P be a polynomial given by a formula of size e. There exist matrices M and N of size $(e+2)\times(e+2)$ such that

$$P = \det M = \operatorname{per} N.$$

Subsequent works

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Subsequent works

- Improved bounds:
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- Extension to weakly-skew circuits, with bound

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  - $e+i+1$: G. Malod & N. Portier [4]

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[2] **Improved construction for universality of determinant and permanent**, Inf. Process. Lett., 2006
[3] **Classes of arithmetic circuits capturing the complexity of computing the determinant**, IEICE T. Inf. Syst., 1992

[4] **Characterizing Valiant's algebraic complexity classes**, J. Compl., 2008

Our results

- Extension to symmetric matrices (characteristic $\neq 2$)

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- Char. 2: Partial permanent is (probably) not VNP-complete

Motivation from Convex Geometry

- Linear Matrix Expression (LME): for A_i symmetric in $\mathbb{R}^{t \times t}$

$$A_0 + x_1 A_1 + \cdots + x_n A_n$$

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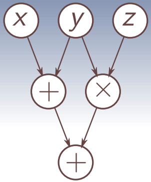
- Drop condition $A_0 \succeq 0 \rightsquigarrow$ exponential size matrices
- What about polynomial size matrices?

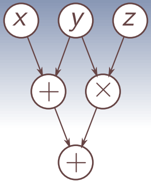
Overview

$(x + y) + (y \times z)$



Circuit: Weakly-skew circuit or formula

Overview



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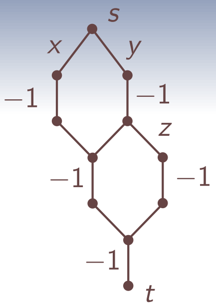
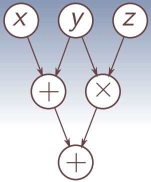
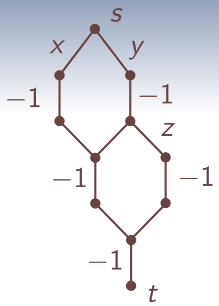




Arithmetic Branching Program

Circuit $\Longrightarrow$ ABP

Overview



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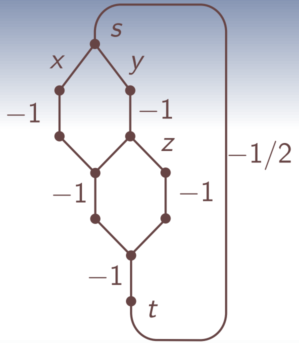
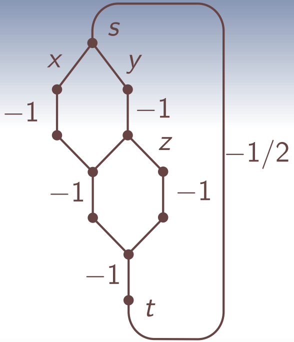
Overview



Circuit $\Longrightarrow$ ABP $\Longrightarrow$ Graph

Overview

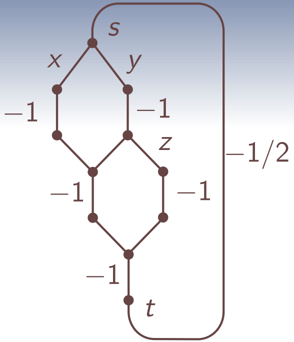


$$\det \begin{vmatrix} 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ x & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{vmatrix}$$

$$= (x + y) + (y \times z)$$

Circuit $\Longrightarrow$ ABP $\Longrightarrow$ Graph $\Longrightarrow$ Matrix

Overview



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$$= (x + y) + (y \times z)$$

Characteristic $\neq 2$

Circuit $\Longrightarrow$ ABP $\Longrightarrow$ Graph $\Longrightarrow$ Matrix

Main new difficulty

Symmetric matrices

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$\implies$ undirected graphs

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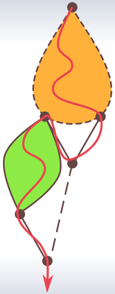

Symmetric matrices
$\Longrightarrow$ undirected graphs
$\Longrightarrow$ "undirected ABPs"

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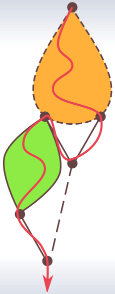
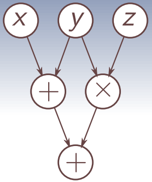


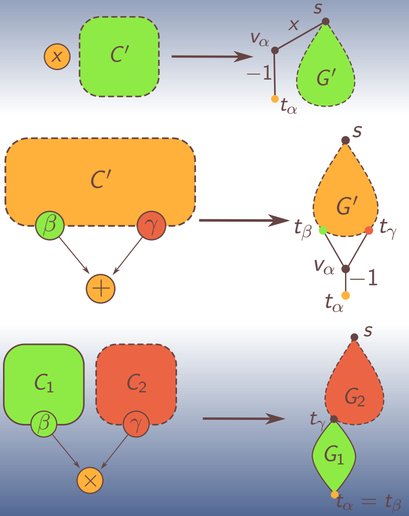
Symmetric matrices
$\Longrightarrow$ undirected graphs
$\Longrightarrow$ "undirected ABPs"

Definition

A path P is acceptable if $G \setminus P$ admits a cycle cover

Weakly-Skew Circuit $\Longrightarrow$ ABP





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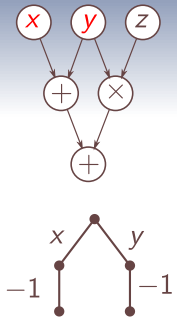

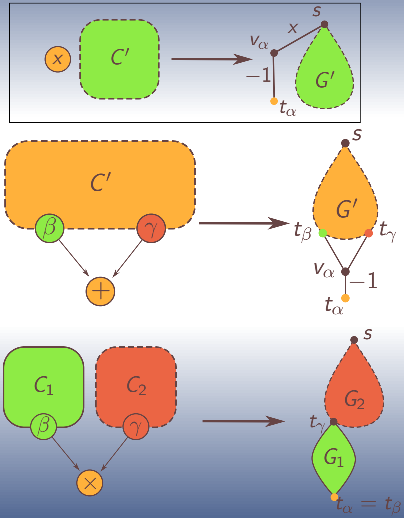

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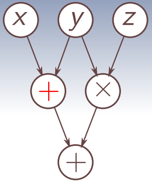

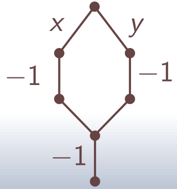

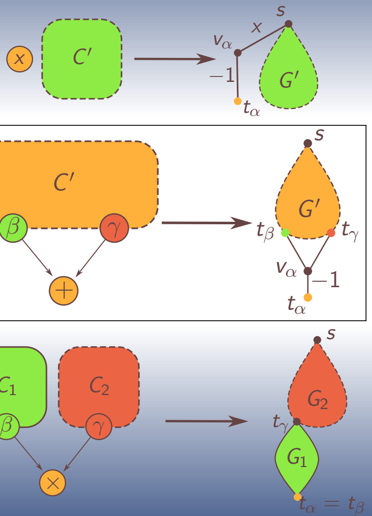

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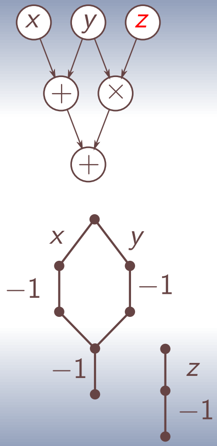
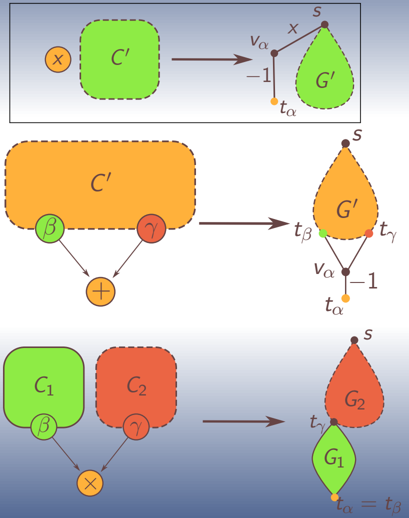
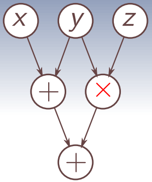
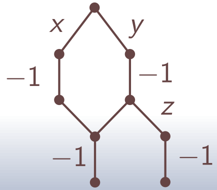
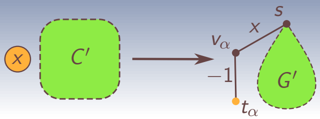


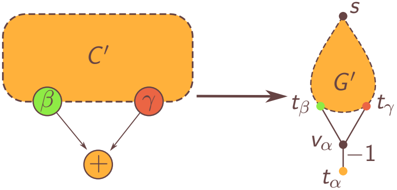


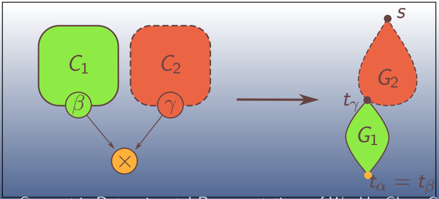
Weakly-Skew Circuit $\implies$ ABP











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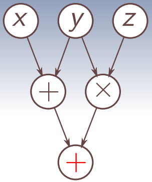
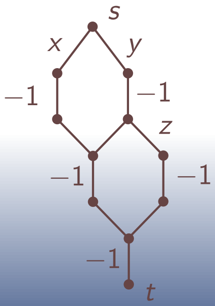
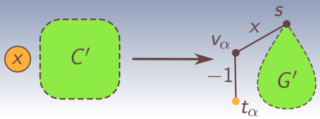
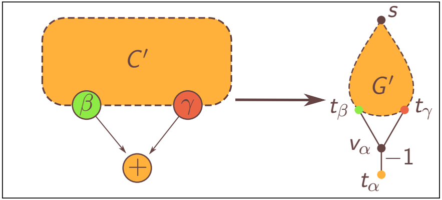
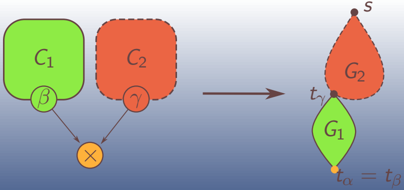


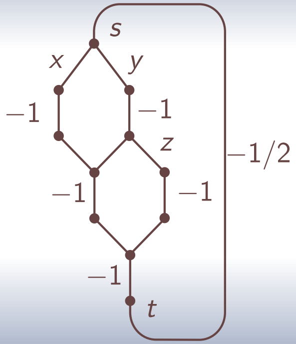








ABP $\Longrightarrow$ Graph



- Add $s \xleftrightarrow{(1/2)\cdot(-1)^{\frac{|G|-1}{2}}} t$: new graph G'.

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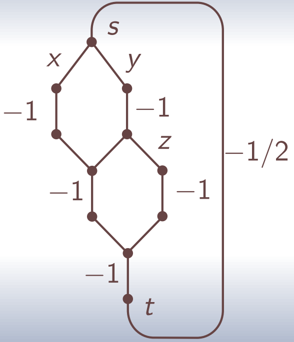

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- Cycle covers of G'

  $\iff$ $s \to t$-paths in G

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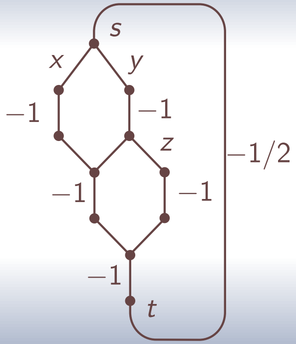

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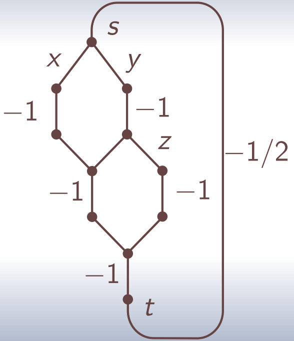
Graph $\Longrightarrow$ Matrix



Determinant

$\mathfrak{S}_n$ = Permutation group of $\{1, \ldots, n\}$

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

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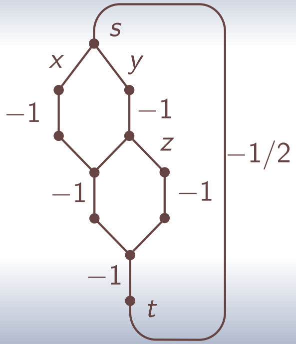

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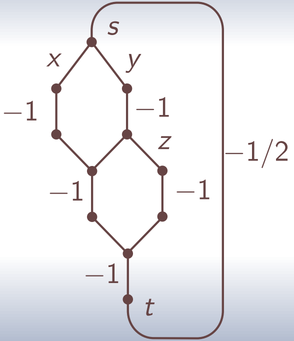

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- permutation in A $\equiv$ cycle cover in G'
- Up to signs, $\det A$ = sum of weights of cycle covers in G'

Summary

$P(x_1, \ldots, x_n)$

Weakly-Skew Circuit

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	Formula	Weakly-skew circuit
Non symmetric	$e+1$	$(e+i)+1$
Symmetric	$2e+1$	$2(e+i)+1$

Problem

Problem [Bürgisser 00]

Is the partial permanent VNP-complete in characteristic 2?

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- VP, VNP, VNP-complete ≡ P, NP, NP-complete for polynomials

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Main lemma

$$(\mathrm{per}^* M)^2 \in \mathrm{VP}$$

A by-product & two updates

Theorem

Let M be an $n \times n$ matrix. Then there exists a symmetric matrix M' of size $O(n^3)$ s.t. $\det M = \det M'$.

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Theorem (Malod)

In characteristic 2, the partial permanent is in VP.

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- Symmetric matrices in Valiant's theory?

Thank you!