Lacunaryx: *Computing bounded-degree factors of lacunary polynomials*

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Factorization of a polynomial f

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$$= (X+Y-1) \times (XY-1) \times (1 + XY + \cdots + X^{568740}Y^{568740})$$

Definition

$$f(X_1,\ldots,X_n)=\sum_{j=1}^{k} c_j X_1^{\alpha_{1j}}\cdots X_n^{\alpha_{nj}}$$

- $\mathrm{size}(f)\simeq k\Big(\max_j(\mathrm{size}(c_j))+n\log(\deg f)\Big)$

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There exist deterministic polynomial-time algorithms computing

- **integer roots** of $f \in \mathbb{Z}[X]$; **[Cucker-Koiran-Smale'98]**
- **low-degree** factors of $f \in \mathbb{Q}(\alpha)[X]$; **[H. Lenstra'99]**
- **low-degree** factors of $f \in \mathbb{Q}(\alpha)[X_1, \dots, X_n]$. **[Kaltofen-Koiran'05-06]**

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Lenstra's algorithm (non-cyclotomic factors)

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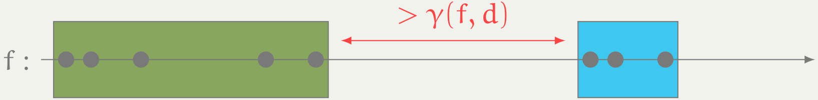

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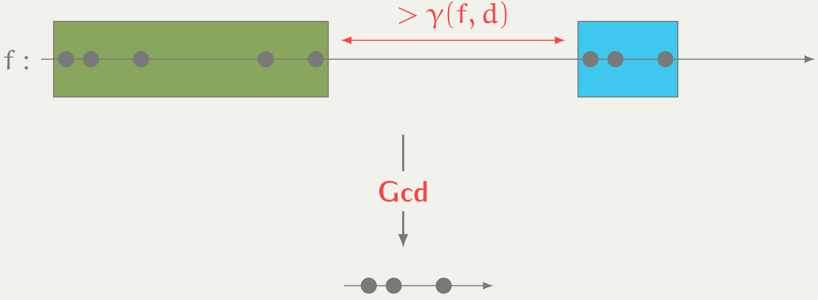

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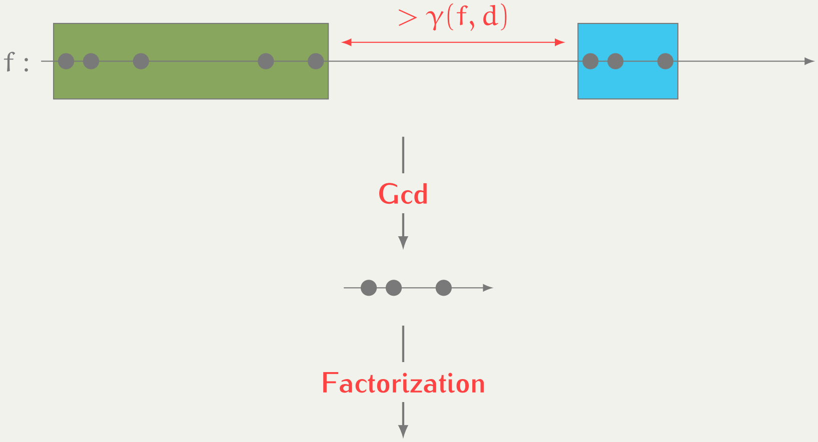

Degree-d non-cyclotomic factors of f

Implemented algorithm (non-cyclotomic factors)

f :

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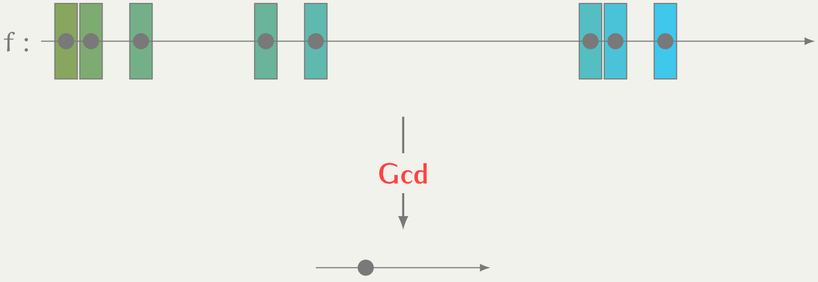

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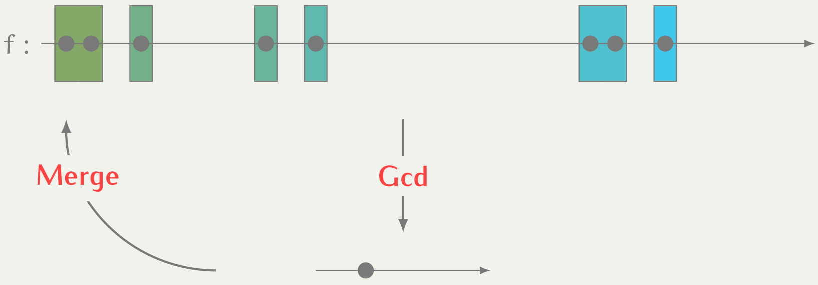

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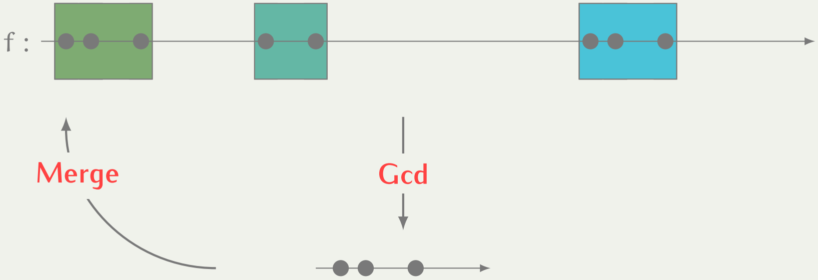

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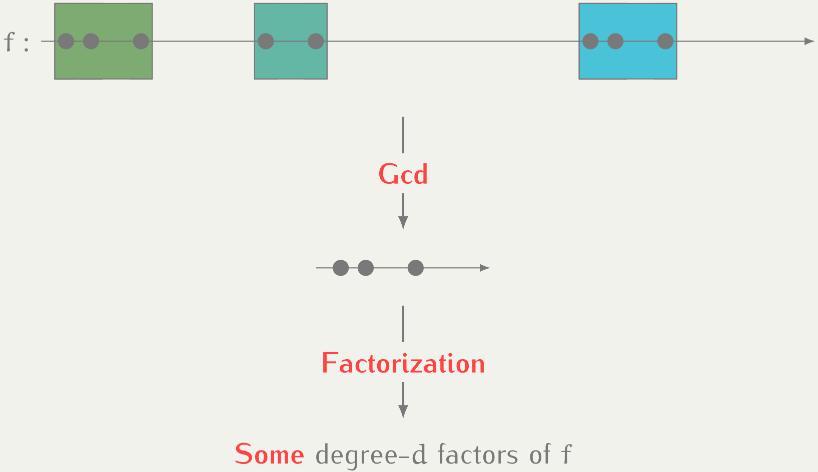

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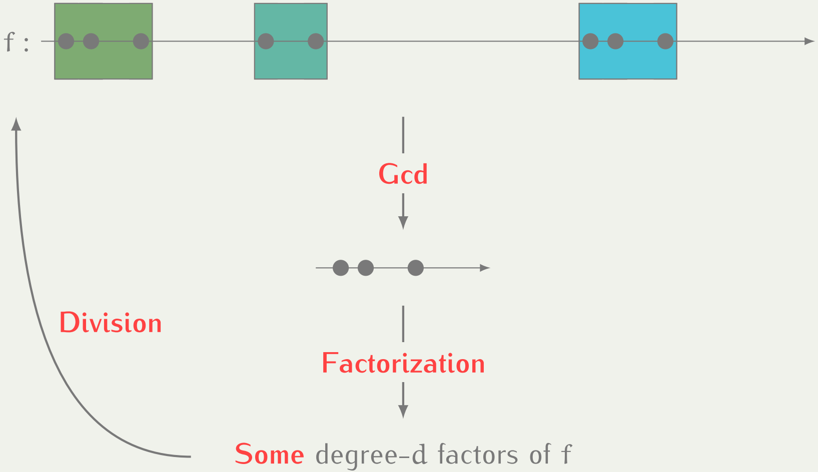

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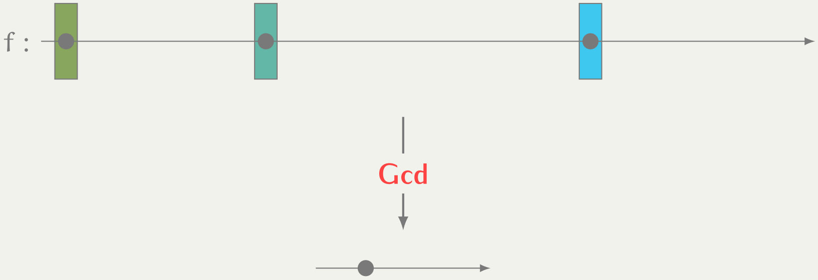

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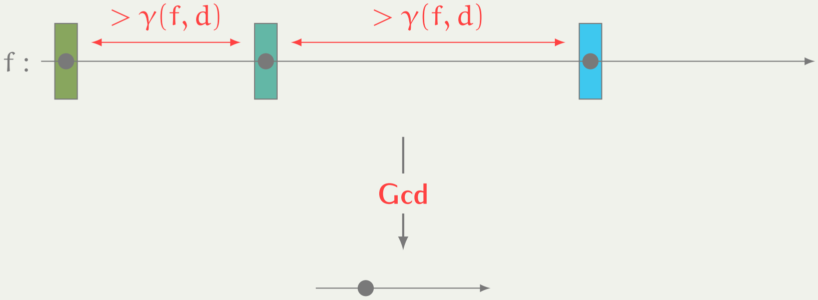

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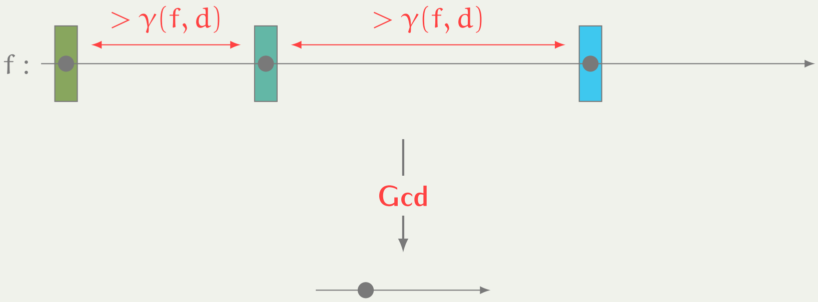

Partial factorization:

$$f = \underbrace{\prod_{g \in G} g}_{\text{low-degree factors}} \times \underbrace{\prod_{h \in H} h}_{\text{sparse polynomials}}$$

- Lenstra's algorithm:
  - ϕ_r divides $f \iff \phi_r$ divides $\gcd(f, X^r - 1)$
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1. For $r = 1$ to $2d^2$, **factor** $\gcd(f, X^r - 1)$; (deg $\phi_r = \varphi(r) \geqslant \sqrt{r/2}$)
2. Return the union of these factors.

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1. Compute $R = \{r : \varphi(r) \leqslant d\}$ (*naive*)
2. For $r \in R$, compute $\phi_r = (X^r - 1)/\gcd(X^r - 1, \prod_{\substack{s \in R \\ s < r}} \phi_s)$
3. Test if ϕ_r **divides** $h^{\bmod r}$

$$f = l \times c \times s$$

- l: product of low-degree polynomials
- c: product of $X^r - 1$
- s: *perturbated* sparse polynomial: $s = \sum_{j=1}^{n} X^{\alpha_j} p_j(X)$

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  5 polynomials, degree $\leqslant 10$, coefficients $\leqslant 50$

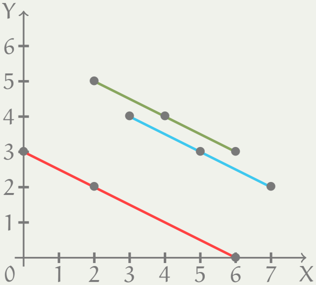
- c: product of $X^r - 1$

  3 polynomials, $r \leqslant 100\,000$

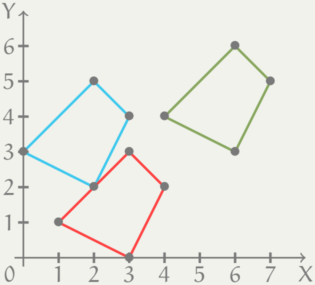
- s: *perturbated* sparse polynomial: $s = \sum_{j=1}^{n} X^{\alpha_j} p_j(X)$

  $n = 40$, $\alpha_j \leqslant 1\,000\,000$, $\deg(p_j) \leqslant 20$, coefficients $\leqslant 50$

$\rightsquigarrow$ degree $\geqslant 1\,000\,000$, $\geqslant 10\,000$ terms, coefficients $\geqslant 5 \times 10^9$



Unidimensional factors



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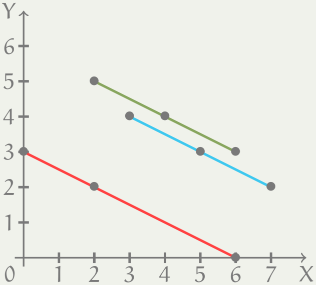
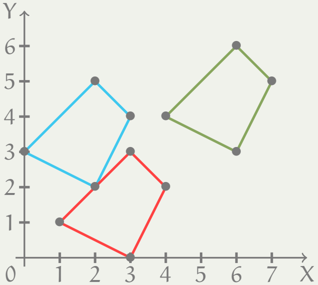


Unidimensional factors

⇓

Univariate lacunary factorization



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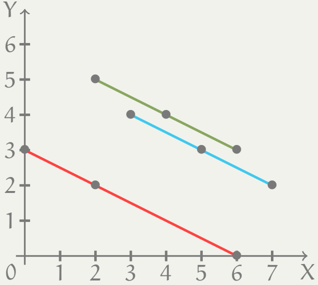
Multivariate low-degree factorization

Unidimensional factors



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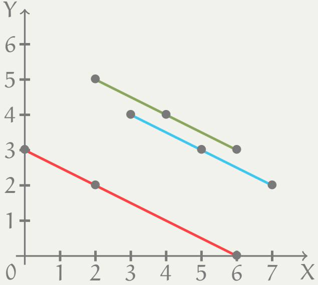
- Each direction δ independently



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1. Write f as a sum of **δ-components**;
2. **Project** each component to a univariate lacunary polynomial;
3. Compute the common **bounded-degree factors** of the projections;
   Compute the factors of the first, then test divisibility
4. **Lift** the univariate factors to unidimensional factors.

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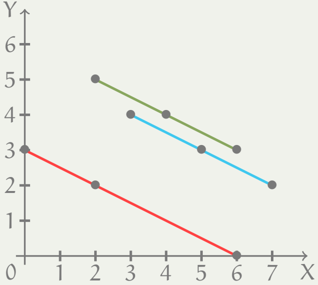

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- Early termination:
  - Degenerate δ-component
  - No common factors

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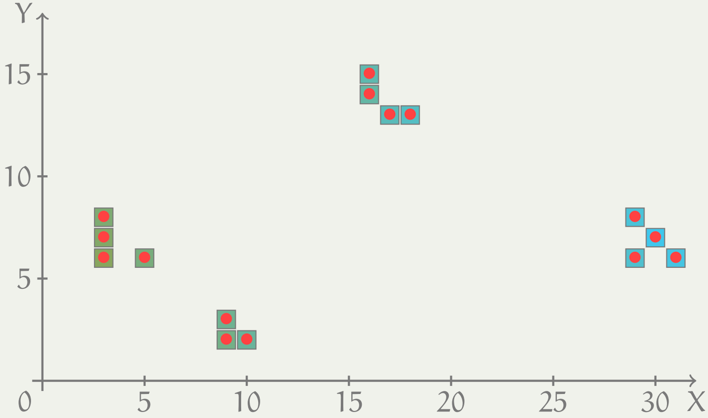

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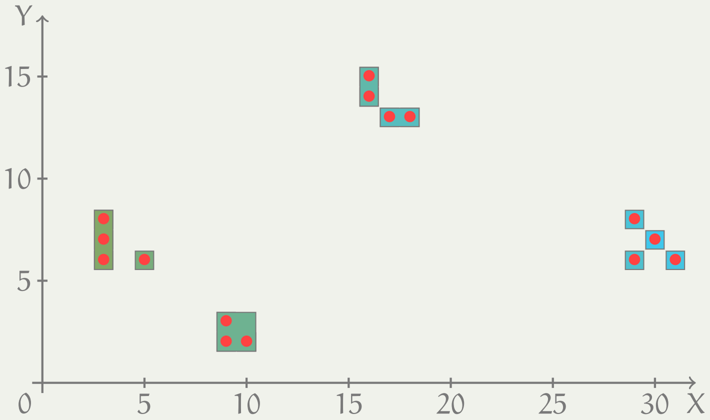

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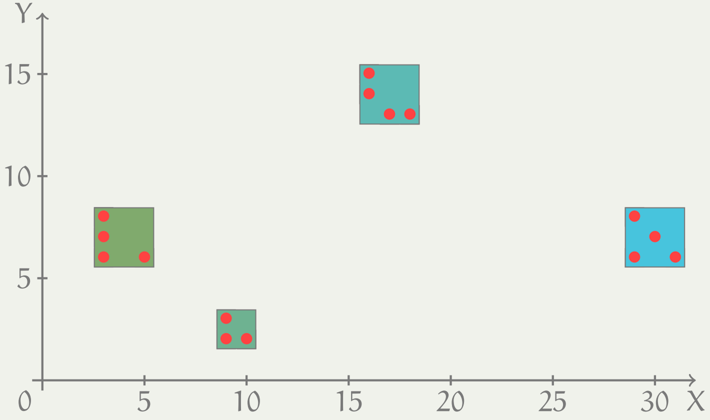

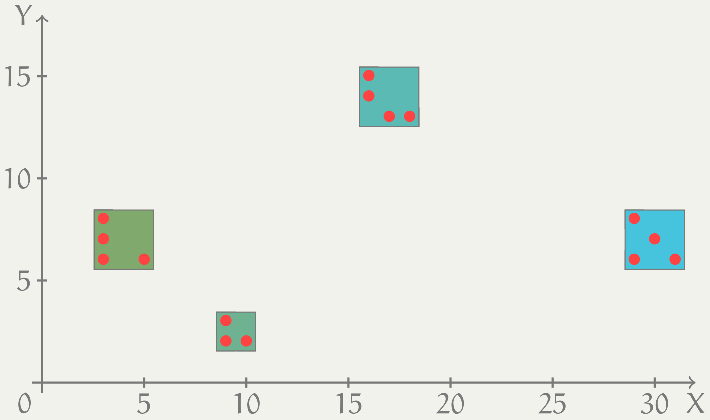

Single-linkage clustering

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Thank you!


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la star !
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