The Multivariate Resultant is NP-hard in any Characteristic

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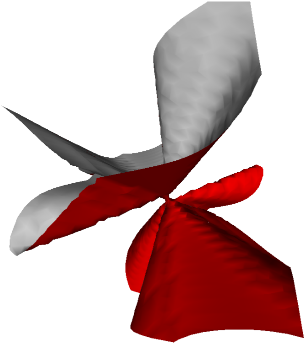
Brno – MFCS 2010 – August 23, 2010

Motivation

- General framework: Resolution of polynomial systems



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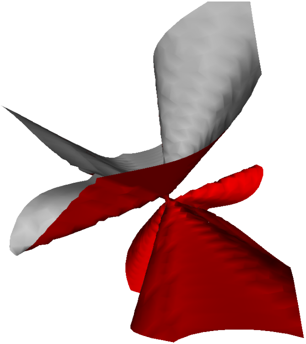

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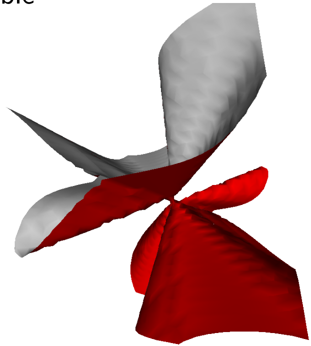

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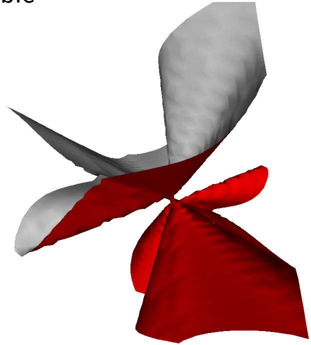

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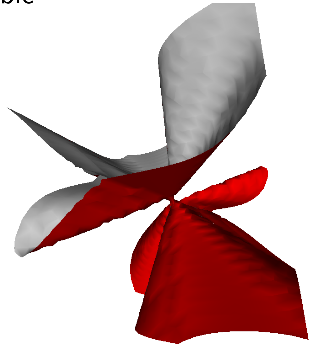

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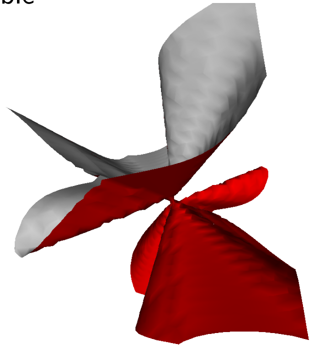

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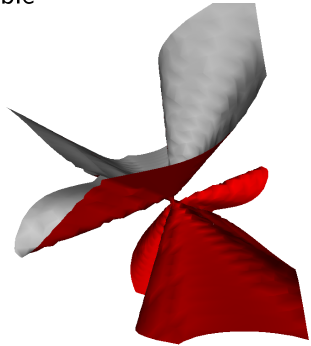

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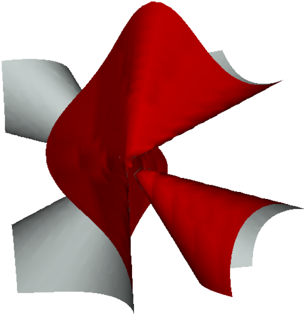
- A few words about Elimination Theory



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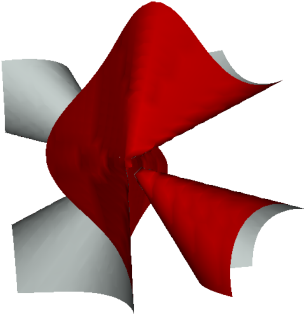

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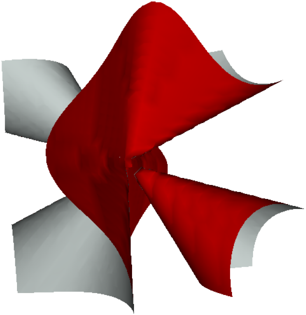
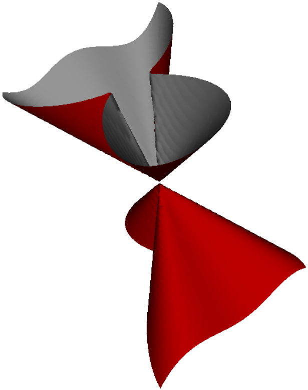
- A few words about Elimination Theory
- First (simple) results about polynomial system solving
- Two ideas to prove NP-hardness



Outline

1. Elimination Theory
2. First results
3. NP-hardness in any characteristic
   - First idea ⇝ randomized reduction
   - Second idea ⇝ deterministic reduction



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There exist $R_1, \dots, R_h \in \mathbb{K}[\bar{\gamma}]$ s.t.

$$\begin{cases} R_1(\bar{\gamma}) & = & 0 \\ & \vdots & \\ R_h(\bar{\gamma}) & = & 0 \end{cases} \Longrightarrow \exists \bar{a}, \quad \begin{cases} f_1(\bar{a}) & = & 0 \\ & \vdots & \\ f_s(\bar{a}) & = & 0 \end{cases}$$

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- Non trivial root?

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- Resultant computable in polynomial space

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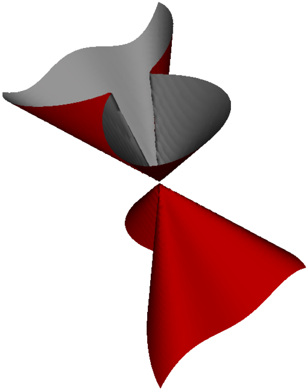

Definitions

Hilbert's Nullstellensatz over $\mathbb{K}$: $\mathtt{HN}(\mathbb{K})$

Input: $f_1, \ldots, f_s \in \mathbb{K}[x_0, \ldots, x_n]$

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- $\mathtt{H_2N}^{\square}(\mathbb{K})$: $s = n + 1$ homogeneous polynomials

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Remark. Best known upper bound for $\mathbb{K} = \mathbb{F}_p$ is PSPACE.

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$\square$

Summary so far

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- NP = AM "under plausible complexity conjectures"

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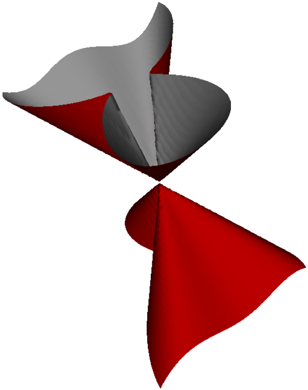

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- Reduction from the case $s > n+1$
  - First idea: decrease the number of polynomials
  - Second idea: increase the number of variables

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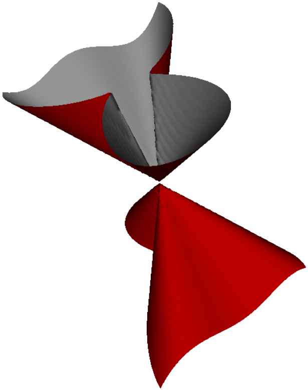

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if random α_{ij}: with high probability

(Quantifier Elimination + Schwartz-Zippel Lemma)

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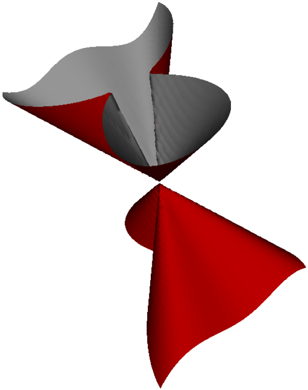

Introduction

- Randomized reduction: fewer polynomials

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  $\neq$ Deterministic reduction: more variables

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$\mathtt{H_2N}$

- Variables over $\mathbb{K}$ x_0 and $x_1, \dots, x_n$
- Polynomials $x_0^2 - x_i^2$ for every i
  - $x_0 \cdot (x_i + x_0)$
  - $x_0 \cdot (x_i + x_j)$
  - $(x_i + x_0)^2 - (x_j + x_0) \cdot (x_k + x_0)$

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$$\left.\begin{array}{l} \blacktriangleright\ x_0 \cdot (x_i + x_0) \\ \blacktriangleright\ x_0 \cdot (x_i + x_j) \\ \blacktriangleright\ (x_i + x_0)^2 - (x_j + x_0) \cdot (x_k + x_0) \end{array}\right\} \quad \to \quad f_{n+1}, \ldots, f_s$$

Reduction

- New variables: $y_1, \ldots, y_{s-n-1}$

New system

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Equivalence?

$(\bar{a}, \bar{b})$: a non trivial root

$$\begin{pmatrix} f_1(\bar{a}) & & \\ \vdots & & \\ f_n(\bar{a}) & & \\ f_{n+1}(\bar{a}) & & +\lambda b_1^2 \\ f_{n+2}(\bar{a}) & -b_1^2 & +\lambda b_2^2 \\ \vdots & & \\ f_{s-1}(\bar{a}) & -b_{s-n-2}^2 & +\lambda b_{s-n-1}^2 \\ f_s(\bar{a}) & -b_{s-n-1}^2 & \end{pmatrix}$$

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$$\det = 0 \overset{?}{\implies} \forall i,\ \epsilon_i = 0 \implies f_1(\bar{a}) = \cdots = f_s(\bar{a}) = 0$$

Last step

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$$\mathtt{H_2N}^{\square}(\mathbb{K}) \text{ is NP-hard}$$

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Thank you!