Symmetric Determinantal Representations of Polynomials

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Motivation from Convex Geometry

- Linear Matrix Expression (LME): for A_i symmetric in $\mathbb{R}^{t\times t}$

$$A_0 + x_1 A_1 + \cdots + x_n A_n$$

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- Applications to Semi-Definite Programming

Valiant (1979)

- Arithmetic formula $\rightsquigarrow$ Determinant

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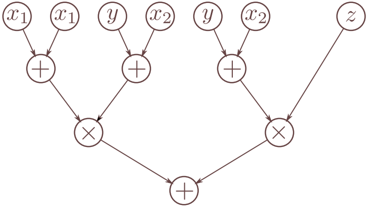

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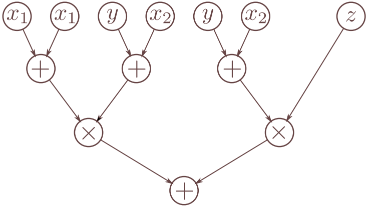

$$\rightsquigarrow \begin{pmatrix} 0 & x_1 & x_1 & 0 & 0 & z & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x_2 & y & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ y & 0 & 0 & 0 & 0 & 1 & x_2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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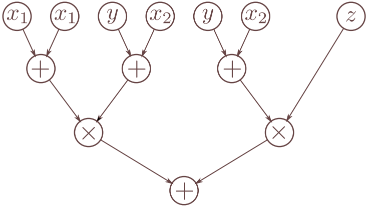

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$$= 2x_1 \cdot (x_2 + y) + z \cdot (x_2 + y)$$

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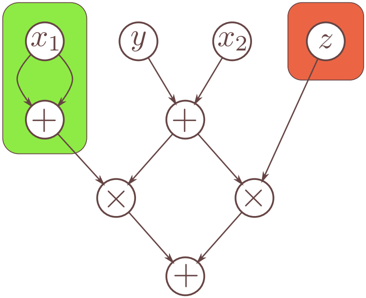

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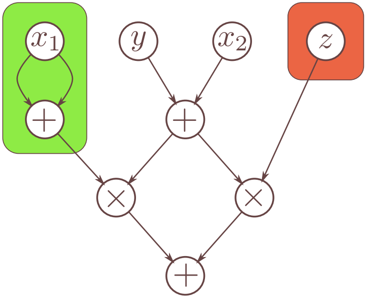

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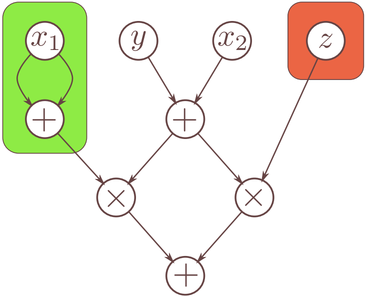

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- Remark: valid for any field

Contents

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- Case of characteristic 2

Outline

Graph-theoretic interpretation of determinants

- Let G be a graph, A its adjacency matrix

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- Up to signs, $\det A$ = sum of the weights of cycle covers in G

Valiant's construction (1/3)

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- In between: a graph G of size $(e+1)$ whose adjacency matrix is A

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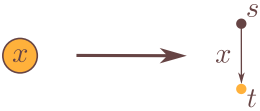

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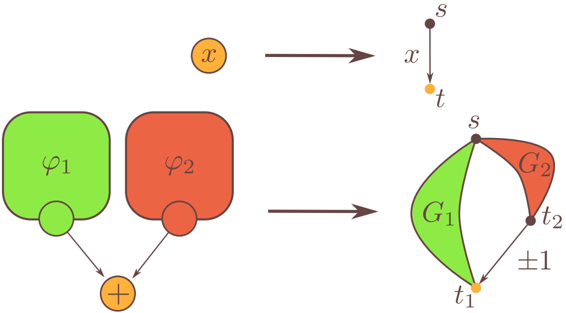

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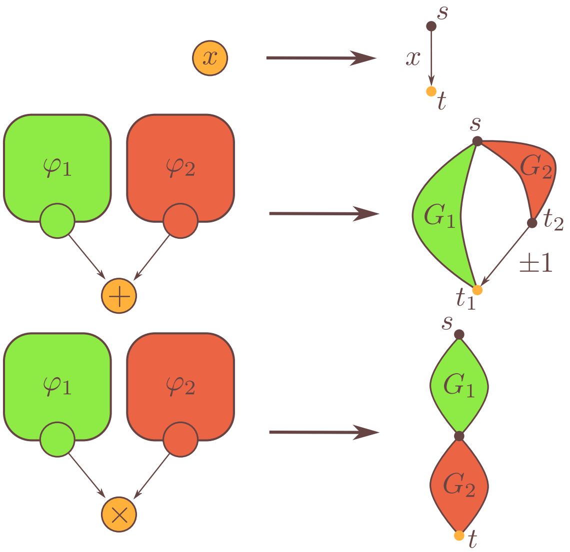

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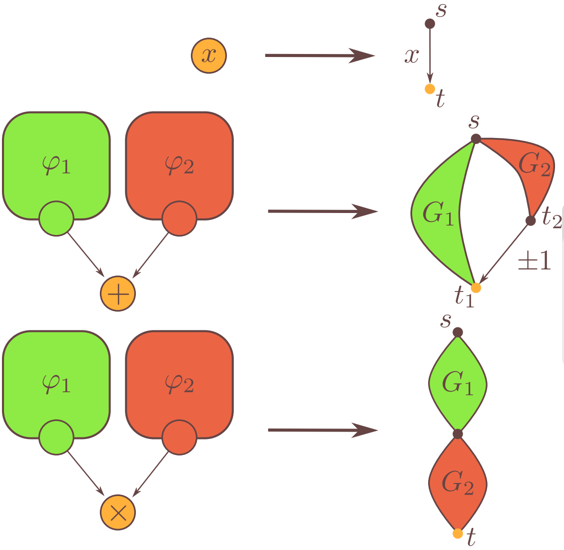

Invariant

$$\varphi = \pm \sum_{s\text{-}t\text{-paths } P} (-1)^{|P|} w(P)$$

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Theorem

For a size-e formula, this construction yields a size-$(e+1)$ graph. Let A be the adjacency matrix of G. Then $\det(A) = \varphi$.

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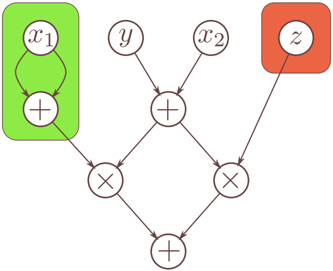

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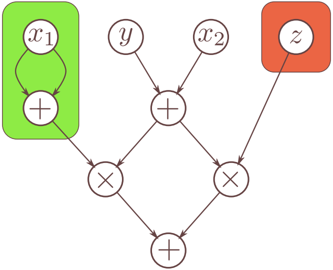

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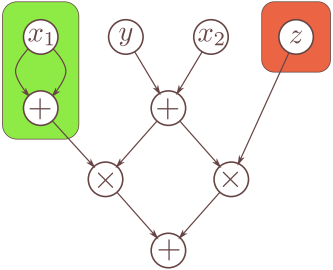

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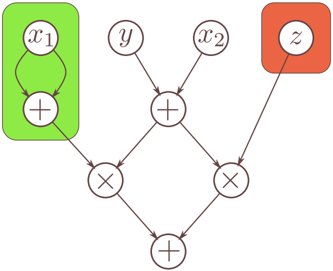

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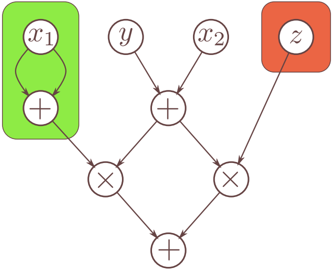

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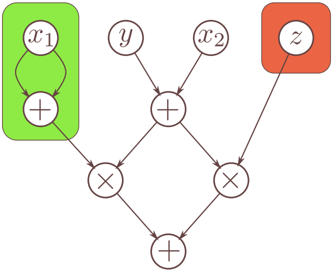

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- Reusable gate: not in a closed subcircuit

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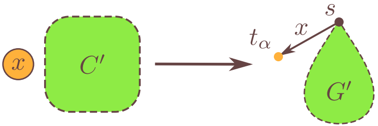

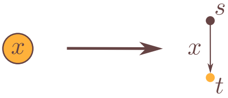

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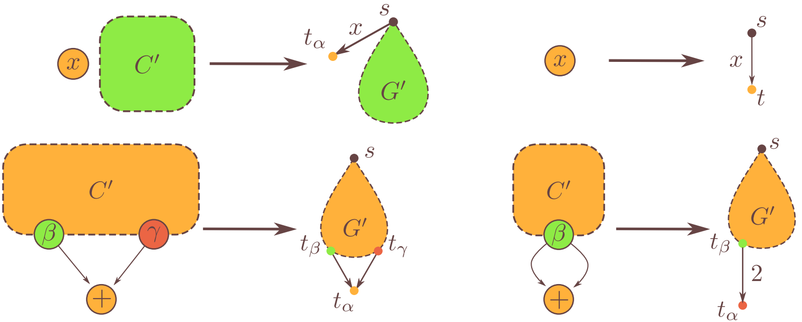

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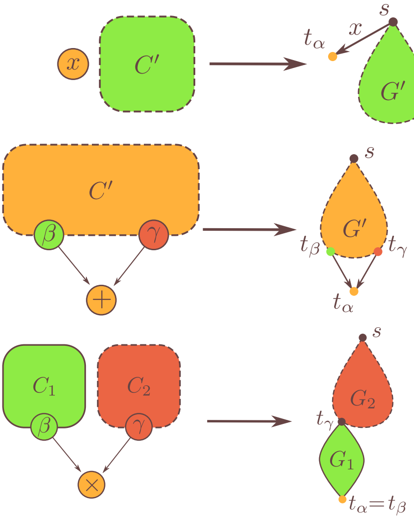

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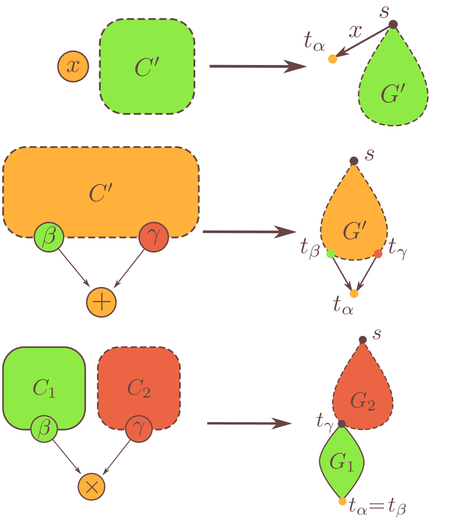

Invariant

For each *reusable* gate α,
there exists t_α s.t.
$w(s \to t_\alpha) = \varphi_\alpha$.

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- N.B.: $\mathrm{char}(\mathbb{K}) \neq 2$ in this section

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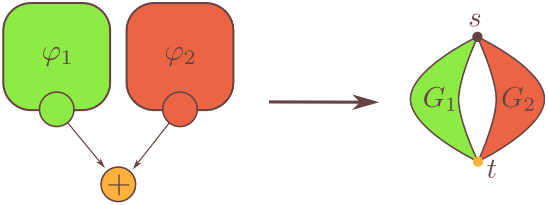

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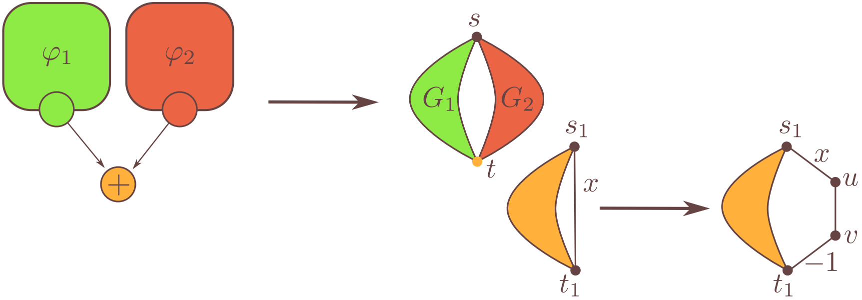

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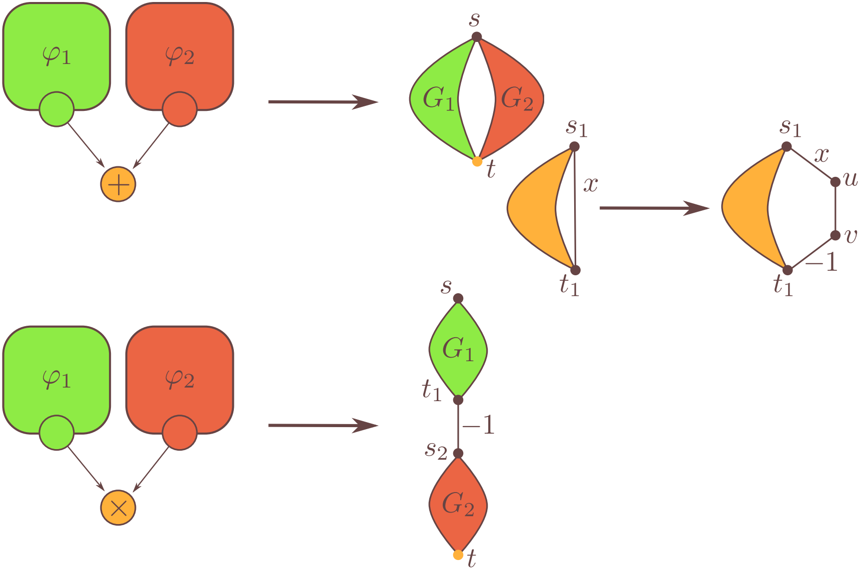

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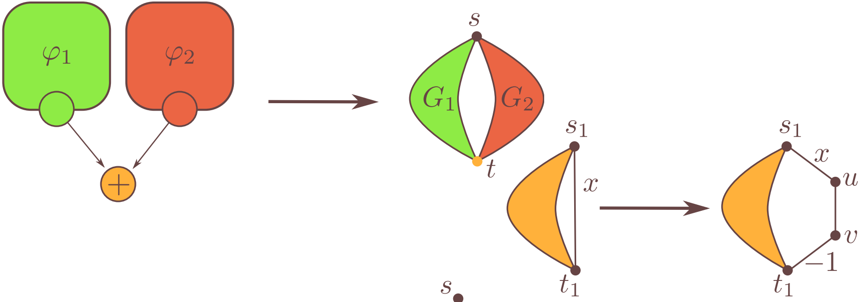

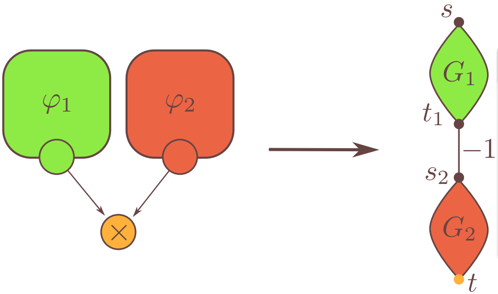

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Invariants for formula's construction

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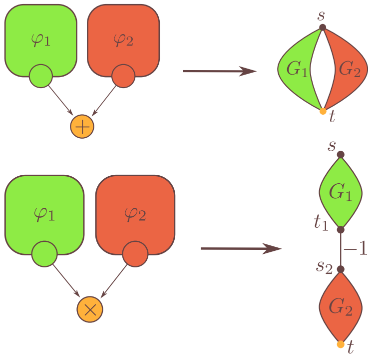

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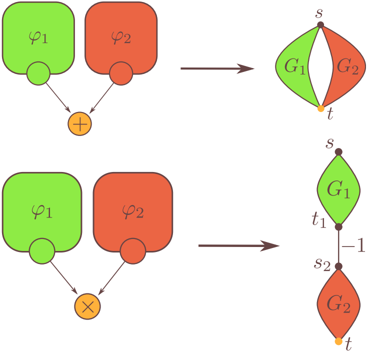

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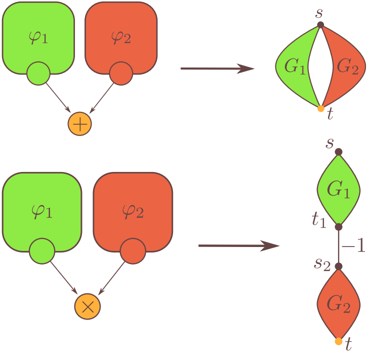

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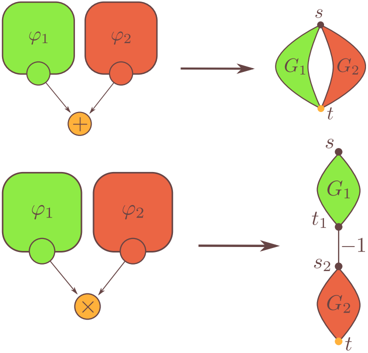

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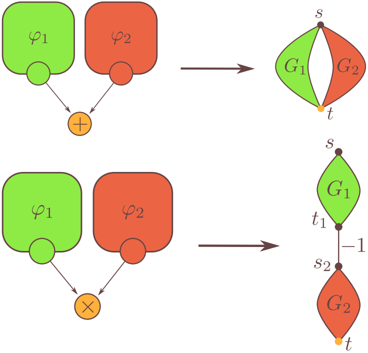

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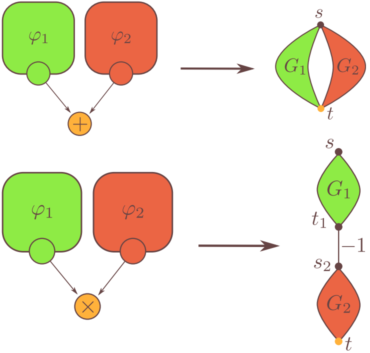
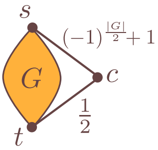


From G to G'



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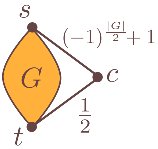

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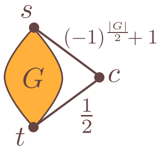

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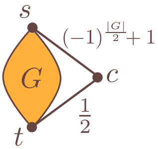

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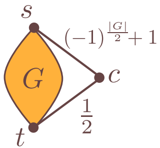

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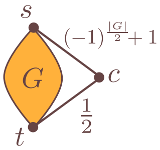

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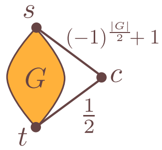

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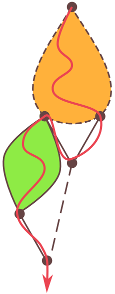
Theorem

For a formula φ of size e, this construction yields a graph of size $2e+3$. The determinant of its adjacency matrix equals φ.

Case of weakly-skew circuits

- Main difficulty:



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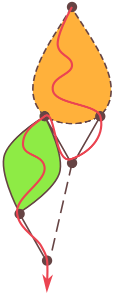
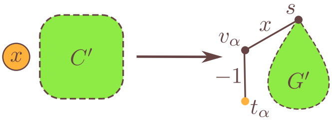


- Definition: an path P is said acceptable if $G \setminus P$ admits a cycle cover

Constructions



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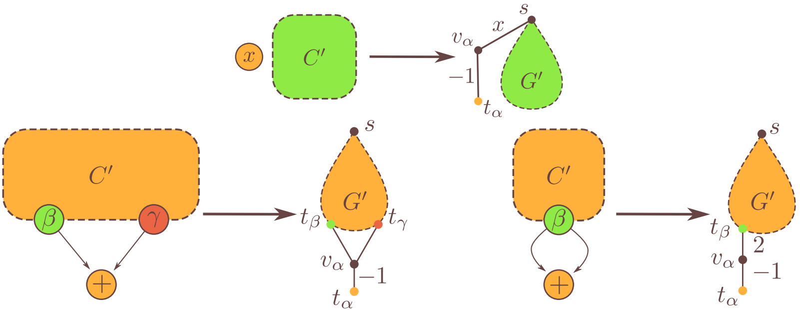

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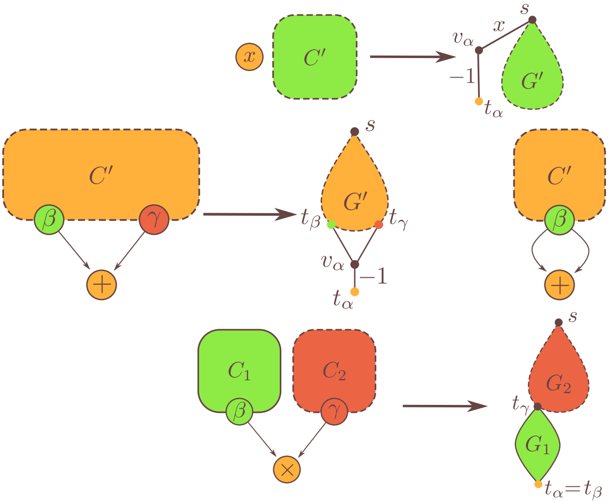

Invariants in the case of weakly-skew circuits

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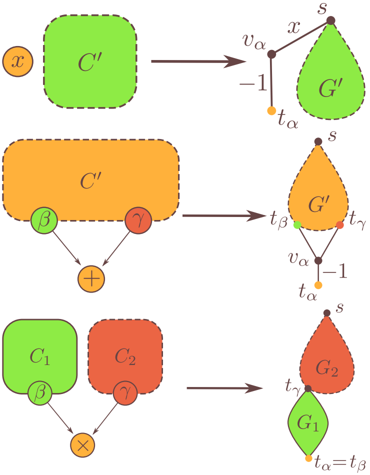

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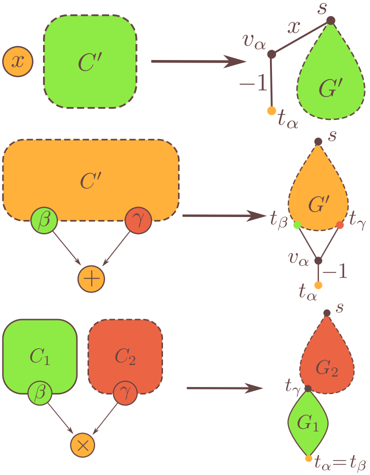

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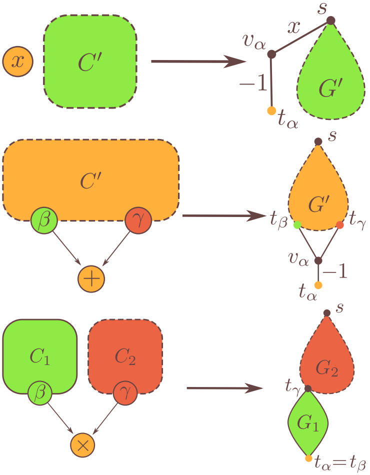

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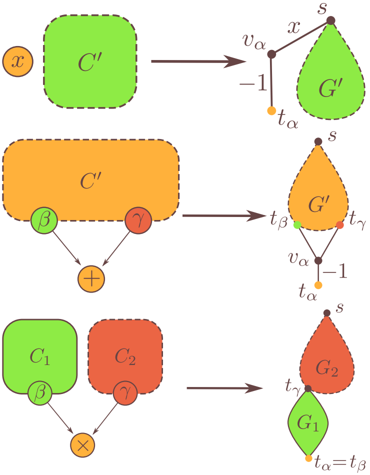

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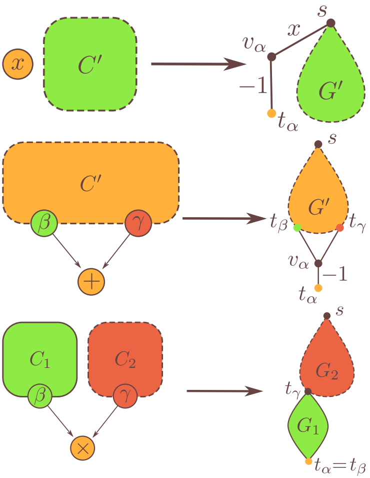

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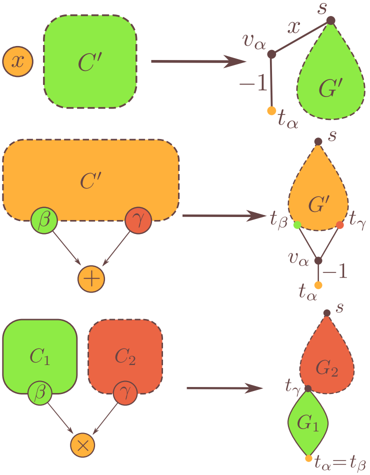

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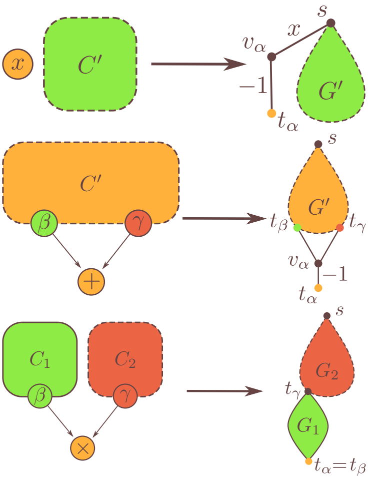

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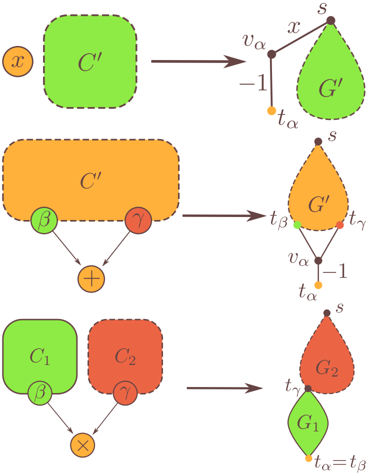

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Theorem

For a weakly skew circuit of size e, with i input variables, computing a polynomial φ, this construction yields a graph G' with $2(e+i)+1$ vertices. The adjacency matrix of G' has its determinant equal to φ.

Outline

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Which polynomials can be represented as determinant of symmetric matrices in characteristic 2?

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- As there is no loop in G', $\det A = \sum_\mu w(\mu)^2 = \left(\sum_\mu w(\mu)\right)^2$

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- $(\mathrm{DET}_n) \in \mathrm{VP}$, $(\mathrm{PER}_n) \in \mathrm{VNP}$, ...

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A family (g_n) is a p-projection of a family (f_n) is there exists a polynomial t s.t. for all n, $g_n(\bar{x}) = f_{t(n)}(a_1, \ldots, a_n)$, with $a_1, \ldots, a_n \in \mathbb{K} \cup \{x_1, \ldots, x_n\}$.

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- (HC_n) is VNP-complete (in any characteristic)

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Same kind of ideas as the previous proof.

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Proof. $((\mathrm{PER}^*)_n^2)$ is a p-projection of (DET_n).

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Proof sketch. If the case arises, $\mathsf{VNP}^2 \subseteq \mathsf{VP}$. This translates into boolean complexity result *via* Bürgisser's boolean parts of Valiant's classes.

A negative result?

Question

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Conjecture

The polynomial $xy + z$ has no such representation

Two-day-old Proof. To do on a board!

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- By-product (in characteristic $\neq 2$):

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- Improvement of Valiant's and Malod's constructions
- Actually, we fixed a flaw in Valiant's 30-year-old proof!
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  - Proof (?) of a negative result (to be verified...)

Future work

- In Convex Geometry: $\mathbb{K} = \mathbb{R}$ and polynomials are *real zero polynomials*.

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- Symmetric matrices in Valiant's theory?

Thank you!