Acceptable Complexity Measures of Theorems

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Historical Overview

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The theorems of a finitely-specified theory cannot be significantly more complex than the theory itself.

- 2005: Calude and Jürgensen prove the “heuristic principle”

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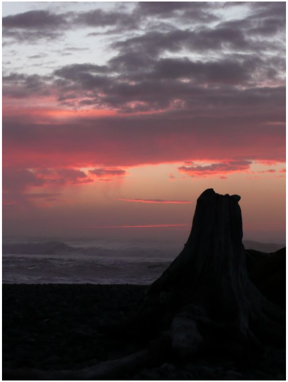

Gillepsie Beach, South Island

Outline

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1. A few definitions
2. About δ
3. Acceptable Complexity Measures
4. An Independence Result
5. Other measures?

Aphabets and strings

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- G: set of all the Gödel numberings

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- Self-delimiting Turing Machine: $PROG_T$ is prefix-free
- Kraft's inequality: for a prefix-free set S, note $r_k = \operatorname{card}\{x \in S : |x|_i = k\}$. Then

$$\sum_{k=1}^{\infty} r_k \cdot i^{-k} \leq 1.$$

Kraft-Chaitin Theorem

Let $(n_k)_{k\in\mathbb{N}}$ be a computable sequence of non-negative integers such that

$$\sum_{k=1}^{\infty} i^{-n_k} \leq 1.$$

Then we can effectively construct a prefix-free sequence of strings $(w_k)_{k\in\mathbb{N}}$ such that for each $k \geq 1$, $|w_k|_i = n_k$.

Program-size complexity

Definition

$$H_{i,T}(x) = \min\left\{|y|_i : y \in X_i^* \text{ and } T(y) = x\right\}$$

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There exists a universal machine U_i such that for every T, there exists c such that

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Definition

x^* is the lexicographically first string of length $H_i(x)$ such that $U_i(x^*) = x$.

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Definition

$$\delta_g(u) = H_2(g(u)) - \lceil \log_2(i) \cdot |x|_i \rceil \, ,$$

where g is a Gödel numbering.

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Theorem

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- For every g and g', there exists a constant d such that

$$\left|H_2(g(u)) - H_2(g'(u))\right| \leq d \text{ and } \left|\delta_g(u) - \delta_{g'}(u)\right| \leq d + 1.$$

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$$\begin{aligned} H_C(g(w)) &\leq |s_{w^*}|_2 = \lceil \log_2(i) \cdot |w^*|_i \rceil = \lceil \log_2(i) \cdot H_i(w) \rceil \\ &\leq \log_2(i) \cdot H_i(w) + 1 \end{aligned}$$

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- If $U_2(w) = g(u)$,

$$H_D(u) \leq \lceil \log_i(2) \cdot |w|_2 \rceil \leq \log_i(2) \cdot |w|_2 + 1 \leq \log_i(2) \cdot H_2(g(u)) + d$$

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Can we improve the bound?

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Proposition

$\forall N > 0,\ \lim_{n\to\infty} i^{-n} \cdot \mathrm{card}\left\{x \in X_i^* : |x|_i = n, \delta_g(x) \leq N\right\} = 0$

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- Definition of a notion of *acceptable* complexity measure
- Properties of those measures
- Which measures are acceptable?

Complexity Measure Builder

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Let $\hat{\rho}_i : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ be a computable function. Then we define the *complexity measure builder* ρ by

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- ρ_g: complexity measure

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The function δ_g is an acceptable complexity measure.

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The program-size complexity is not an acceptable complexity measure.

(i) ✗ $\mathrm{card}\{x \in X_i^* : H_2(g(x)) \leq N\} \leq 2^N$

(ii) ✓ $\{x \in X_i^* : |x|_i = n, H_2(g(x)) \leq N\} = \emptyset$ for large enough n

(iii) ✓ Already seen as a corollary.

Outline

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- Study of two complexity builders, not acceptable.

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- Independence of the three conditions in the definition.

First example

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- We can use the results about δ_g.

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(i) ✓ The bound is always valid.

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Proposition

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- If (i) holds, $\mathrm{card}\,\{x \in \mathcal{T} : |x| = n\} \leq \alpha \cdot n^{\beta \cdot N_{\mathcal{F}}}$.
- There is an exponential number of provable formulae like

$$\forall x_1 \exists x_2 \exists x_3 \ldots \forall x_k \bigwedge_{l=1}^{k} (x_l = x_l).$$

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Theorem

The three conditions are independent from each other.

Proof of the independence (1)

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(iii) ✗ Else, (*) is false.

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- ρ^2 satisfies (ii) and (iii) but not (i).
- Either (iii) is always satisfied, or δ^2 satisfies (i) and (ii) but not (iii).

Outline

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Proposition

Suppose that ρ_g is acceptable. Then so is $\alpha \cdot \rho_g + \beta$, $\alpha, \beta \in \mathbb{Q}$, $\alpha > 0$.

Linear variations of the program-size complexity

Proposition

Let $\hat{\rho}_i : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}$ be a computable function, linear in both variables. If it defines an acceptable complexity measure, then

$$\hat{\rho}_i(x, y) = a \cdot (x - \varepsilon \cdot \lceil \log_2(i) \cdot y \rceil) + b,$$

where $1/2 \leq \varepsilon \leq 1$.

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- Between $1/2$ and 1, your ideas are welcome!

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The End
Thank you very much!