The Limited Power of Powering

Polynomial Identity Testing and a Depth-four Lower Bound for the Permanent

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Representation of Univariate Polynomials

$$P(X) = X^{10} + 5X^6 + 3X^2 + 1$$

Representations

- Dense: $[1, 0, 0, 0, 5, 0, 0, 0, 3, 0, 1]$
- Sparse: $\{(10, 1), (6, 5), (2, 3), (0, 1)\}$

Representation of Multivariate Polynomials

$$P(x, y, z) = x^5 y^3 z^2 + 5xy^4 z + 3yz + 1$$

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Sparse representation not always relevant either.

Arithmetic Circuits

$$
\begin{aligned}
Q(x,y,z) = {} & x^4 + 4\,x^3y + 6\,x^2y^2 + 4\,xy^3 + x^2z + 2\,xyz \\
& + y^2z + x^2 + y^4 + 2\,xy + y^2 + z^2 + 2\,z + 1
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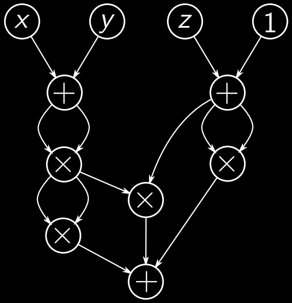

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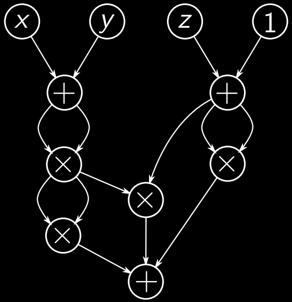

$\rightsquigarrow$ Straight Line Programs

Algebraic Complexity Theory

Complexity of a polynomial = size of its smallest circuit

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  - Polynomial complexity: Determinant

$$\det\left((x_{ij})_{1\le i,j\le n}\right) = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\prod_{i=1}^{n} x_{i\sigma(i)}$$

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Complexity of a polynomial = size of its smallest circuit

- Which polynomials have low/high complexity?
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Conjecture (Algebraic P $\neq$ NP)

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- (Boolean) Complexity of problems on circuits
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  - Roots finding, factorization, …

Permanent & Polynomial Identity Testing

- PIT: randomized polynomial-time algorithm

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$\leadsto$ Connections between PIT and lower bounds already in [Heintz-Schnorr'80]

The τ-conjecture

Conjecture (Shub & Smale, 1995)

For any $f \in \mathbb{Z}[X]$ of complexity $\tau(f)$,

$$\#\{n \in \mathbb{Z} : f(n) = 0\} \leq \mathrm{poly}(\tau(f)).$$

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Sum of products of sparse polynomials

Definition

Let $\mathrm{SPS}(k, m, t, A)$ the class of polynomials

$$f(X) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j(X)^{\alpha_{ij}}$$

where the $f_j \in \mathbb{R}[X]$ are t-sparse and $0 \leq \alpha_{ij} \leq A$.

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- Known techniques: $2^{\mathcal{O}((kmt)^2)}$ [Khovanskii'80, Risler'85]

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Conjecture (Koiran, 2011)

Let $f \in \mathrm{SPS}(k, m, t, A)$, then

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1. Upper bound on # real roots of $f \in \mathrm{SPS}(k, m, t, A)$
2. Lower bound for the permanent
3. Links with Polynomial Identity Testing

Upper bound for the number of real roots of SPS polynomials

Theorem

There exists $C > 0$ such that the number of real roots of any $f = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}} \in \mathrm{SPS}(k, m, t, A)$ is at most

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- Independent of A.
- If k and m are fixed, this is polynomial in t.

Case $k = 2$

Proposition

The polynomial

$$f = \prod_{j=1}^{m} f_j^{\alpha_j} + \prod_{j=1}^{m} f_j^{\beta_j}$$

has at most $2mt^m + 4m(t-1)$ *real roots.*

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$$F' = \underbrace{\prod_{j=1}^{m} f_j^{\beta_j - \alpha_j - 1}}_{\leq 2m(t-1) \text{ roots and poles}} \times \underbrace{\sum_{j=1}^{m} (\beta_j - \alpha_j) f_j' \prod_{l \neq j} f_l}_{\leq 2mt^m - 1 \text{ roots}}.$$

The permanent family

$$\mathrm{PER}_n(x_{11}, \ldots, x_{nn}) = \operatorname{per} \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} x_{i\sigma(i)}$$

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$n \mapsto \tau(\mathrm{PER}_n)$ grows faster than any polynomial function.

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- The conjecture for depth-4 circuits implies the general case [Agrawal–Vinay'08, Koiran'11]

Multivariate SPS polynomials

Definition

$(P_n)_{n\geq 0} \in \mathrm{mSPS}(k,m)$ if

$$P_n(x_1,\dots,x_{Q(n)}) = \sum_{i=1}^{k}\prod_{j=1}^{m} f_{j,n}^{\alpha_{ij,n}}(\vec{x})$$

where

- $f_{j,n}$ is $Q(n)$-sparse;

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- exponential-size depth-4 circuits
- polynomial-size circuits with polynomial-depth

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Theorem

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Proof sketch. $(\mathrm{PER}_n) \in \mathrm{mSPS}(k, m)$

$$\implies \mathrm{PW}_n(X) = \prod_{i=1}^{2^n}(X - i) \in \mathrm{SPS}(k, m, \mathrm{poly}(n), 2^{\mathrm{poly}(n)})$$

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But PW_n has 2^n roots: contradiction.

Links with PIT

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For fixed k and m, we can test for zero $f \in \mathrm{SPS}(k, m, t, A)$ in time polynomial in t and A.

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Remark. Works also with mSPS polynomials (Kronecker substitution).

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Full version: arXiv:1107.1434