Factorization of lacunary polynomials

Bruno Grenet

LIRMM — Université de Montpellier

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Classical polynomial arithmetic

Representations

- Univariate polynomials: list of coefficients (size ∼ degree)
- Multivariate polynomials: list of coefficients (size: $\binom{n+d}{n}$) or monomials (size ∼ number of monomials)

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Efficient algorithms over $\mathbb{Z}$, $\mathbb{Q}(\alpha)$, $\overline{\mathbb{Q}}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_q$, …

- Multiplication, division with remainder, gcd, evaluation / interpolation — poly(deg.) (or even quasi-linear!)
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Lacunary polynomials

$$
\begin{aligned}
& \left(X^{543615} - 12X^{451234} + 4\right) \times \left(3X^{27653} - 6X^{8765} - 17\right) \\
= \quad & 3X^{571268} - 6X^{552380} - 17X^{543615} - 36X^{478887} \\
& + 72X^{459999} + 204X^{451234} + 12X^{27653} - 24X^{8765} - 68
\end{aligned}
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Definition

$$f(X_1, \ldots, X_n) = \sum_{j=1}^{k} c_j X_1^{\alpha_{1j}} \cdots X_n^{\alpha_{nj}}$$

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- Multiplication $f \times g$: $\mathcal{O}(k_f \times k_g)$ $\rightsquigarrow$ $\mathcal{O}(k_f + k_g + k_{f \times g})$?

Algorithms for lacunary polynomials

Bad news

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Good news

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- Computing roots of $f \in \mathbb{F}_p[X]$ is NP-hard **[Bi-Cheng-Rojas'13]**

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Lacunary factorization algorithm

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$$= (X + Y - 1) \times (X^{568741}Y^{568741} - 1)$$
$$= (X + Y - 1) \times (XY - 1) \times (1 + XY + \cdots + X^{568740}Y^{568740})$$

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Theorems

There exist deterministic polynomial-time algorithms computing

- **integer roots** of $f \in \mathbb{Z}[X]$; [Cucker-Koiran-Smale'98]
- **low-degree** factors of $f \in \mathbb{Q}(\alpha)[X]$; [Lenstra'99]
- **low-degree** factors of $f \in \mathbb{Q}(\alpha)[X_1, \dots, X_n]$. [Kaltofen-Koiran'06, G.'14]

It is NP-**hard** to compute **roots of** $f \in \mathbb{F}_p[X]$. [Bi-Cheng-Rojas'13]

Univariate polynomials

over $\mathbb{Z}$, $\mathbb{Q}$ or a number field

Integer roots of integral polynomials

Theorem [Cucker-Koiran-Smale'98]

There exists a deterministic **polynomial-time** algorithm to compute the integer roots of a **lacunary polynomial** $f \in \mathbb{Z}[X]$.

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Let $f = f_1 + f_2 \in \mathbb{Z}[X]$, with coefficients of absolute value $\leqslant 2^s$, s.t. $\mathrm{val}(f_2) - \deg(f_1) > 1 + s$. Then for $|x| \geqslant 2$, $f(x) = 0 \implies f_1(x) = f_2(x) = 0$.

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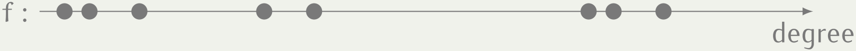
$$-9 + X^2 + 6X^7 + 2X^8 = -9 + X^2 + X^7(6 + 2X)$$

Lenstra's algorithm (non-cyclotomic factors)



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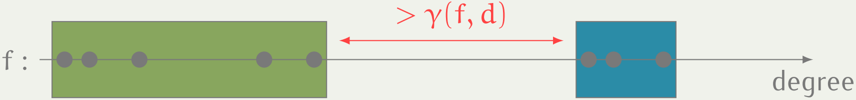

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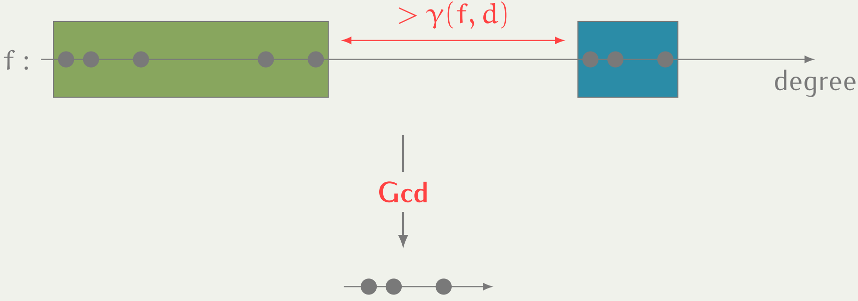

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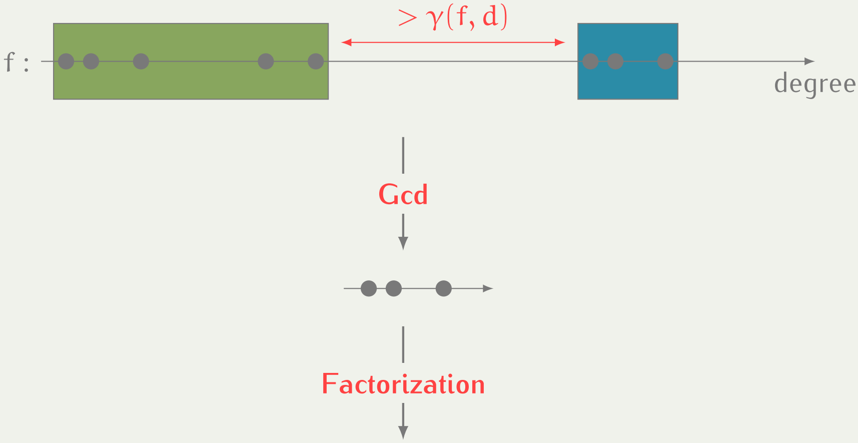

f :

$> \gamma(f, d)$

degree

Gcd

Factorization

Degree-d non-cyclotomic factors of f

Implemented algorithm (non-cyclotomic factors)

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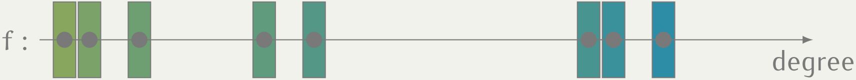

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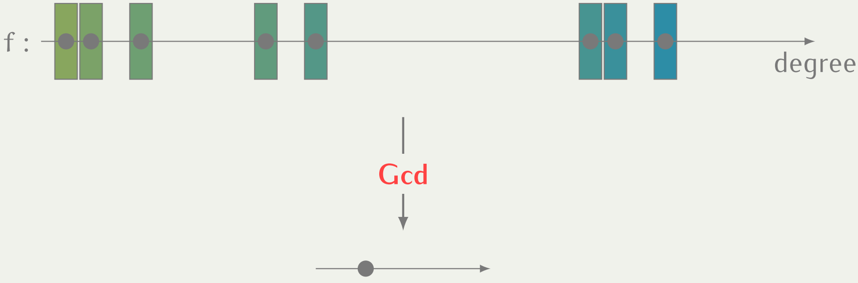

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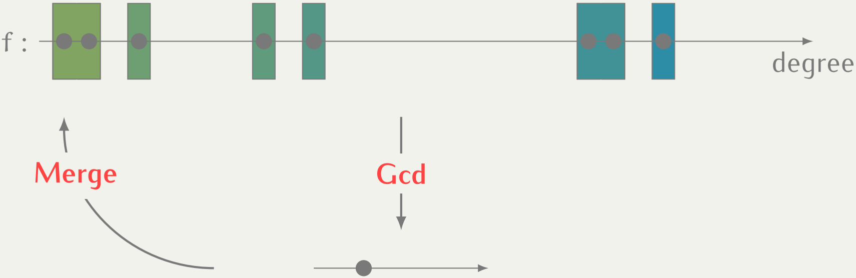

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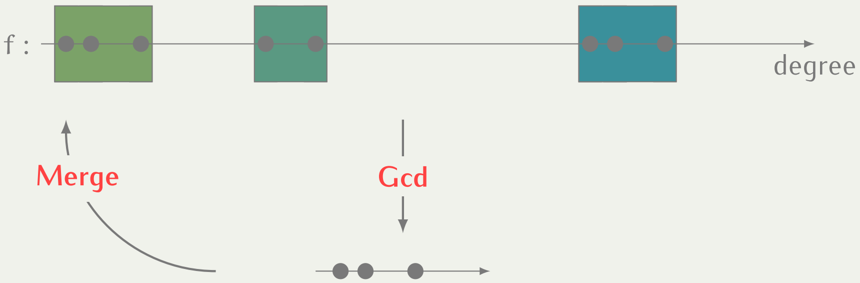

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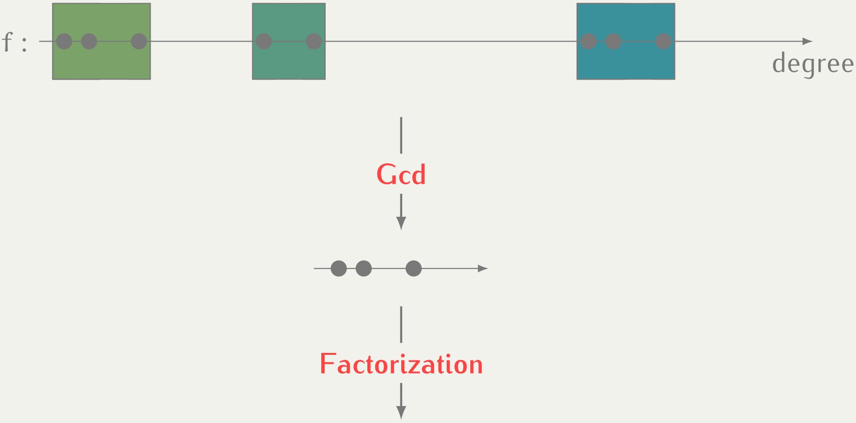

f :

degree

Gcd

Factorization

Some degree-d factors of f

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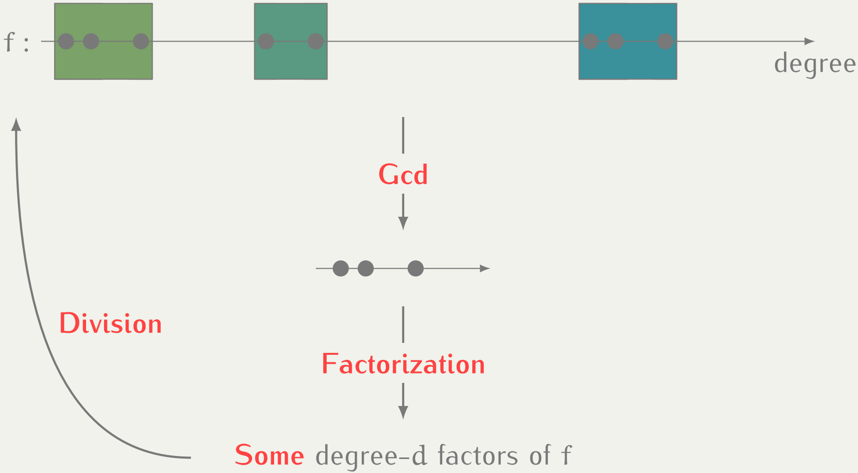

f :

degree

Gcd

Factorization

Division

Some degree-d factors of f

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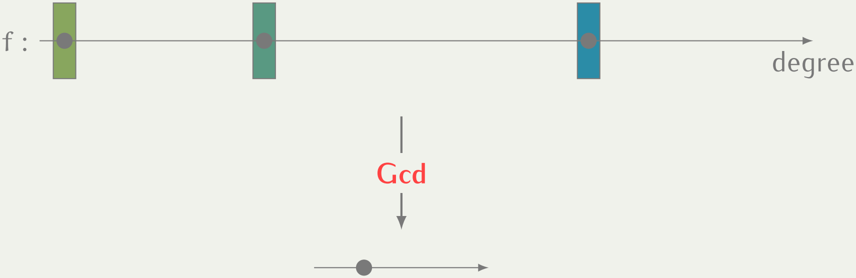
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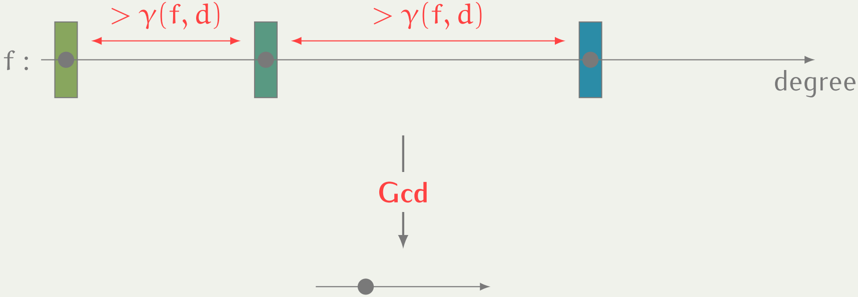

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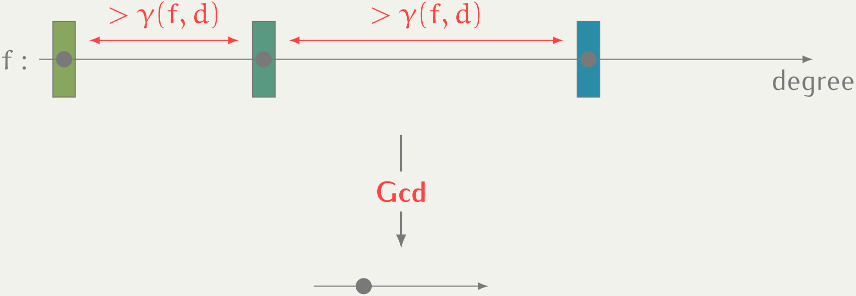

Partial factorization:

$$f = \underbrace{\prod_{g\in G} g}_{\text{low-degree factors}} \times \underbrace{\prod_{h\in H} h}_{\text{sparse polynomials}}$$

Example

Package Lacunaryx of Mathemagix

$$f = l \times c \times s$$

- l: product of low-degree polynomials
- c: product of $X^r - 1$
- s: *perturbated* sparse polynomial: $s = \sum_{j=1}^{n} X^{\alpha_j} p_j(X)$

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⇝ **degree $\geqslant 1\,000\,000$, $\geqslant 10\,000$ terms, coefficients $\geqslant 5 \times 10^9$**

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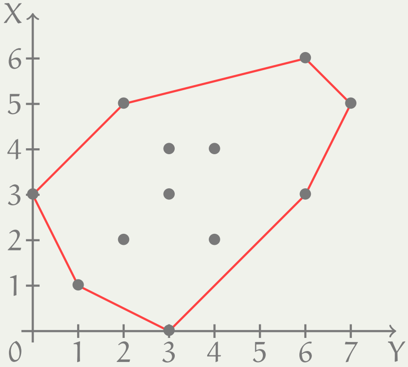
degree	5	10	15	20
time (ms)	1994	2924	12190	26165

Intel© Core™ CPU @ 2.60GHz with 7.7GB RAM

Multivariate polynomials

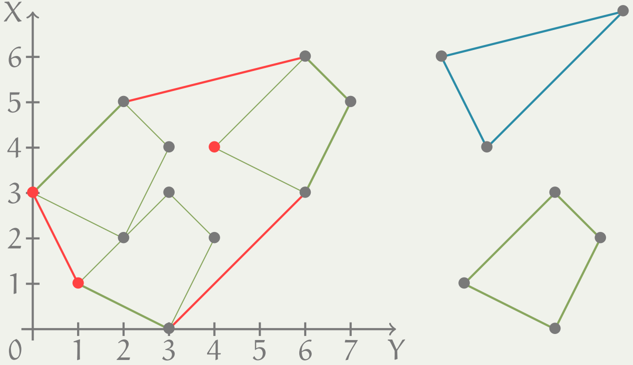
over any field of characteristic 0

Newton polygon/tope



$$f = Y^3 + 2XY - X^2Y^4 + X^3Y^3 - 2X^2Y^2 - 4X^3 + 2X^4Y^3 - 2X^5Y^2 + X^3Y^6 + 2X^4Y^4 - X^5Y^7 + X^6Y^6$$

Newton polygon/tope



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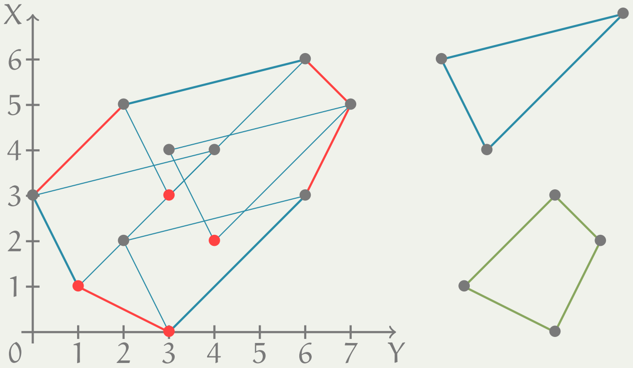
Newton polygon/tope



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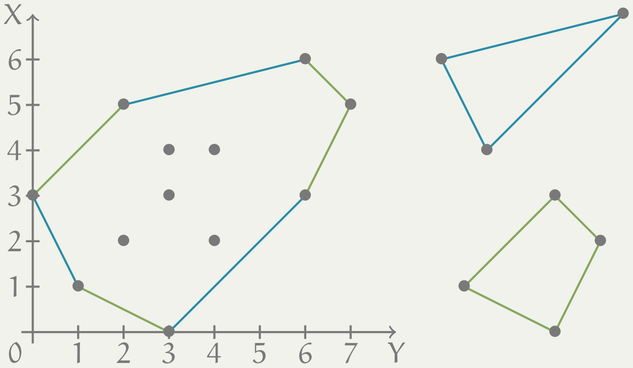

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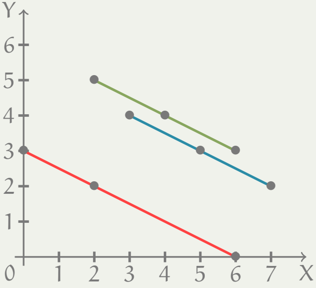
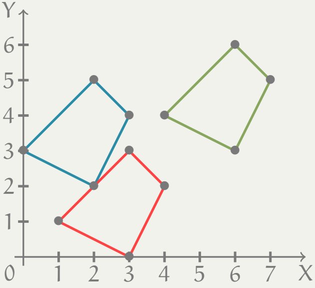
Two kinds of factors

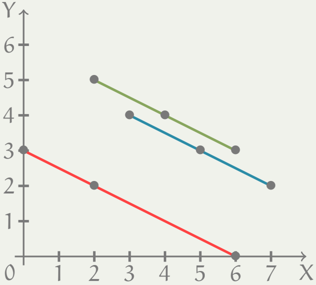


Unidimensional factors



Multidimensional factors

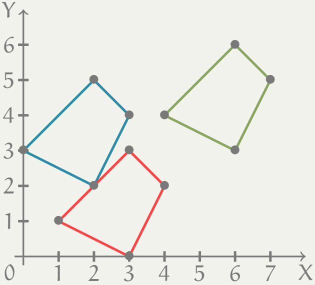
Two kinds of factors



Unidimensional factors

⇓

Univariate lacunary factorization

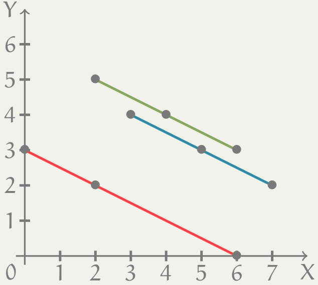


Multidimensional factors

⇓

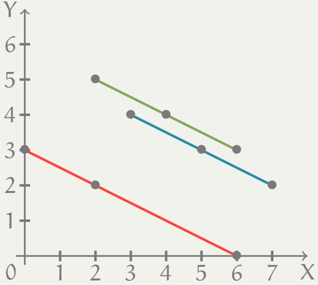
Multivariate low-degree factorization

Unidimensional factors



- $X^{\gamma} f(X^{\delta})$ for some univariate $f \in \mathbb{K}[Z]$
- Each direction δ independently

Unidimensional factors



- $X^\gamma f(X^\delta)$ for some univariate $f \in \mathbb{K}[Z]$
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1. Write f as a sum of **δ-components**
2. **Project** each component to a univariate lacunary polynomial
3. Compute the common **bounded-degree factors** of the projections
   Available for number field only. NP-hard in positive characteristic.
4. **Lift** the univariate factors to unidimensional factors

Linear factors of bivariate polynomials

[Chattopadhyay-G.-Koiran-Portier-Strozecki'13]

Observation

$(Y - uX - v)$ divides $f(X, Y) \iff f(X, uX + v) \equiv 0$

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Theorem

$$\operatorname{val}\left(\sum_{j=1}^{\ell} c_j X^{\alpha_j} (uX + v)^{\beta_j}\right) \leqslant \alpha_1 + \binom{\ell}{2} \text{ if nonzero and } uv \neq 0.$$

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Gap Theorem

Suppose that $f = f_1 + f_2$ with $\mathrm{val}_X(f_2) - \mathrm{val}_X(f_1) > \binom{s(f_1)}{2}$. Then for all $uv \neq 0$, $(Y - uX - v)$ divides f iff it divides both f_1 and f_2.

An example

$$\begin{aligned} f = X^{31}Y^{6} - 2X^{30}Y^{7} + X^{29}Y^{8} - X^{29}Y^{6} + X^{18}Y^{13} \\ - X^{16}Y^{15} + X^{17}Y^{13} + X^{16}Y^{14} + X^{10}Y^{2} - X^{9}Y^{3} \\ + X^{9}Y^{2} - X^{5}Y^{6} + X^{3}Y^{8} - 2X^{3}Y^{7} + X^{3}Y^{6} \end{aligned}$$

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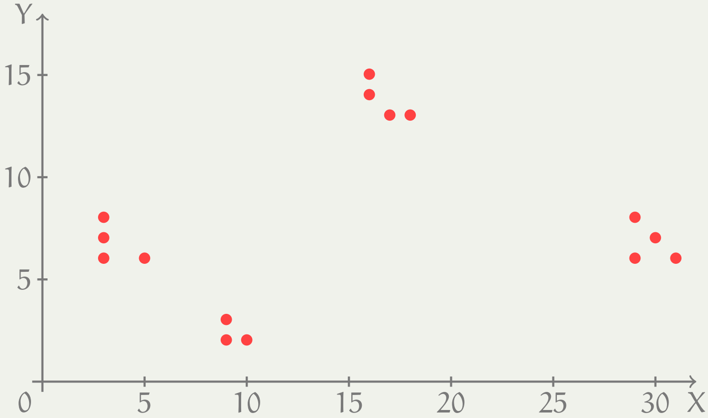

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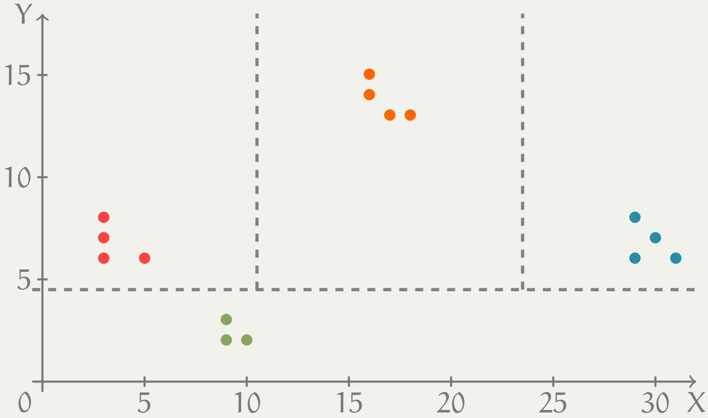
$$+ X^{9}Y^{2} - X^{5}Y^{6} + X^{3}Y^{8} - 2X^{3}Y^{7} + X^{3}Y^{6}$$



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$$f_1 = X^3Y^6(X - Y + 1)(1 - X - Y)$$

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$\Longrightarrow$ linear factors of f: $(X - Y + 1, 1)$, $(X, 3)$, $(Y, 2)$

Observation for low-degree factors

$g(X,Y)$ divides $f(X,Y) \iff f(X,\phi(X)) \equiv 0$

Puiseux series

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- $\phi_1,\ldots,\phi_d$ are **Puiseux series**:

$$\phi(X) = \sum_{t\geqslant t_0} a_t X^{t/n} \text{ with } a_t \in \overline{\mathbb{K}},\ a_{t_0} \neq 0. \qquad (\mathrm{val}(\phi) = t_0/n)$$

Valuation bound

Theorem [G.'14]

$$\operatorname{val}\left(\sum_{j=1}^{\ell} c_j X^{\alpha_j} \phi(X)^{\beta_j}\right) \leqslant \min_j(\alpha_j + \nu\beta_j) + (8d^2 - \nu)\binom{\ell}{2},$$

where

- $\phi \in \overline{\mathbb{K}}\langle\langle X \rangle\rangle$ of valuation ν and degree-d minimal polynomial,
- the family $(X^{\alpha_j}\phi^{\beta_j})_j$ is linearly independent.

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- Bounds $\alpha_j + \nu\beta_j$, not the degree $\rightsquigarrow$ use two different ν's
- Multivariate polynomials: recursive use of the Gap Theorem

Algorithm

Corollary

Inputs: $f \in \mathbb{K}[X_1, \ldots, X_n]$ in lacunary representation
bound d

Output: Degree-$O(d^4 s(f)^2)$ polynomial f_{ld} s.t for all multidimensional degree-d polynomial g, $\text{mult}_g(f) = \text{mult}_g(f_{\text{ld}})$

Complexity: Deterministic polynomial time

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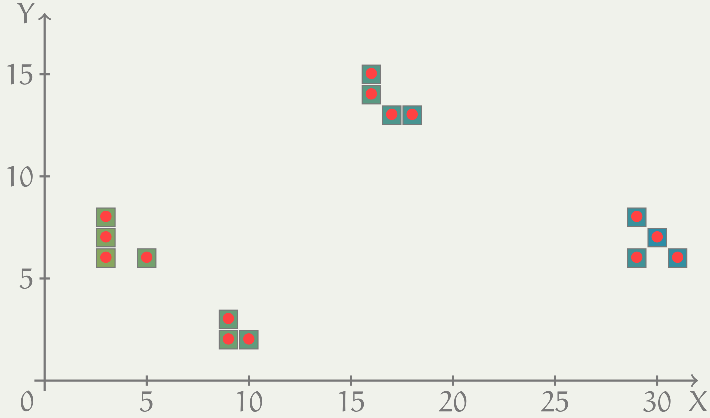
Complexity: Deterministic polynomial time

1. Write $f = f_1 + \cdots + f_s$ where $\deg_{X_i}(f_t) - \mathrm{val}_{X_i}(f_t) \leqslant (4d^4 - 2d^2)\binom{s(f_t)}{2}$ for all i;
2. Return $\gcd(f_1, \dots, f_s)$.

(3. Factor the gcd using a low-degree factorization algorithm.)

Implemented algorithm



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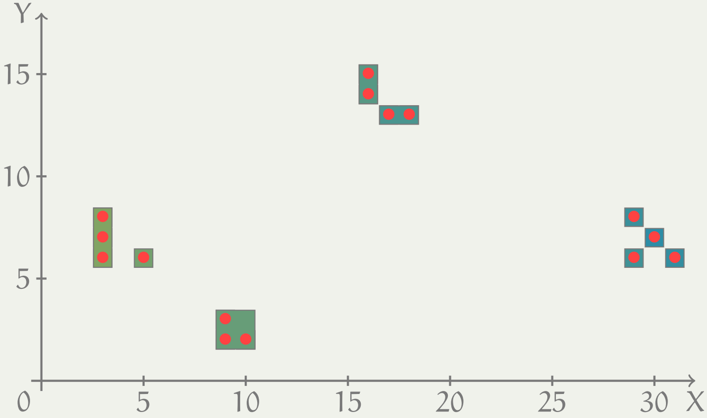

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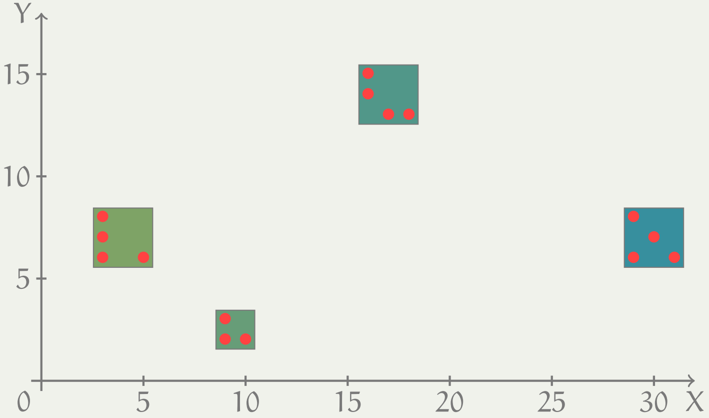

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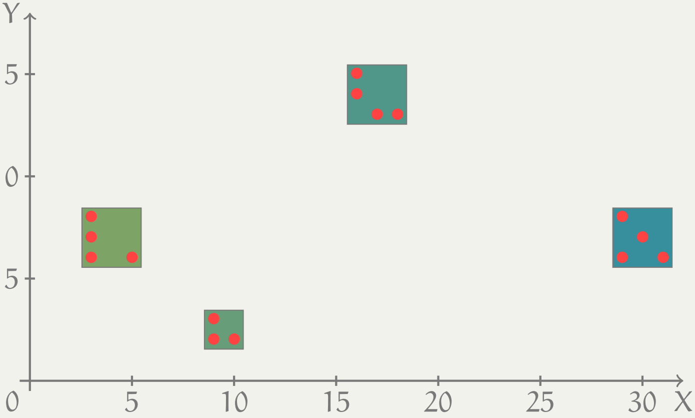
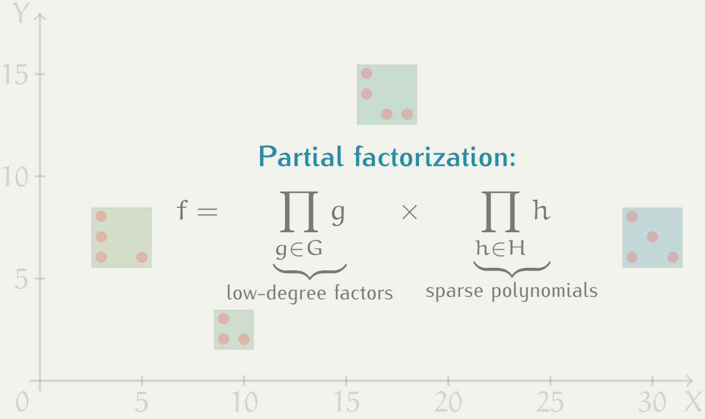


Single-linkage clustering

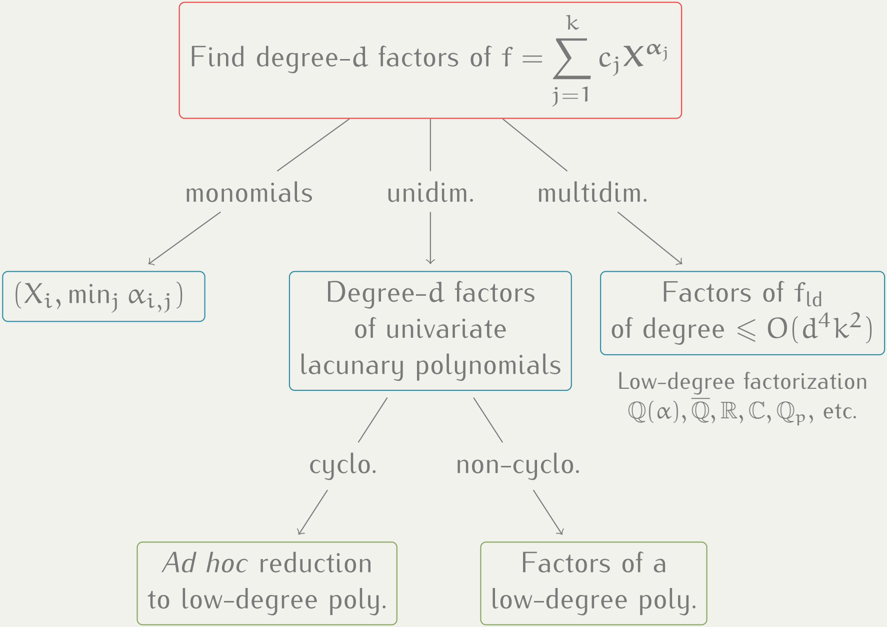
Implemented algorithm



Partial factorization:

$$f = \underbrace{\prod_{g\in G} g}_{\text{low-degree factors}} \times \underbrace{\prod_{h\in H} h}_{\text{sparse polynomials}}$$

Single-linkage clustering



Find degree-d factors of $f = \sum_{j=1}^{k} c_j X^{\alpha_j}$

monomials

unidim.

multidim.

$(X_i, \min_j \alpha_{i,j})$

Degree-d factors of univariate lacunary polynomials

Factors of f_{ld} of degree $\leqslant O(d^4 k^2)$

Low-degree factorization $\mathbb{Q}(\alpha), \overline{\mathbb{Q}}, \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$, etc.

cyclo.

non-cyclo.

Ad hoc reduction to low-degree poly.

Factors of a low-degree poly.

Available for $\mathbb{Q}(\alpha)$ only. Impossible for $\overline{\mathbb{Q}}$, $\mathbb{C}$

```
Mmx] use "lacunaryx"; x : LPolynomial Integer == lpolynomial(1,1);
     p == x^3*(x-2)*(2*x+3)^2*(-x+3)*(2*x+7)*(x^2+x+1)*(3*x+5);
     q == x^3 - 6 - 2*x^4 + 12*x + x^5 - 6*x^2+ 3*x^1345 - 6*x^1346 + 3*x^1347 +
          8*x^432534 - 18*x^432535 + 12*x^432536 - 2*x^432537 + 1 - 2*x + x^2;
     e : Integer == 35154014504040115230143514;
     r == 1 + 3*x^1345 - 2*(x-4)*x^e + (x^3-6)*x^(2*e);
     pqr == p*q*r; (log deg pqr/log 2, #pqr)
```

$(85.861891823199, 149)$

```
                                                                        49 msec
```

```
Mmx] roots pqr
```

$[[2, 1], [3, 1], [0, 3], [1, 2]]$

```
                                                                        43 msec
```

```
Mmx] X == coordinate ('x); x : LMVPolynomial Integer == lmvpolynomial(1, X);
     Y == coordinate ('y); y : LMVPolynomial Integer == lmvpolynomial(1, Y);
     f == x^2*y*(x-2)*(2*y+3)^2*(y-x+3)*(2*x+7*y)*(x*y+x+1)*(3*x-6*y+5);
     g == x^3*y^54354165 - 6*y^54354165 - 2*x^4*y^54354164 + 12*x*y^54354164
     + x^5*y^54354163 - 6*x^2*y^54354163 + 3*x^1345*y^54336 - 6*x^1346*y^54335
     + 3*x^1347*y^54334 + 8*x^432534*y^5 - 18*x^432535*y^4 + 12*x^432536*y^3 -
     2*x^432537*y^2 + y^2 - 2*x*y + x^2;
     h == 1 + 3*x^1345*y^54334 - 2*(x-4*y)*x^e*y^2 + (x^3-6)*y^(2*e);
     fgh == f*g*h; (log deg fgh/log 2, #fgh)
```

$(85.861891823199, 1028)$

```
                                                                        60 msec
```

```
Mmx] linear_factors fgh
```

$[[x, 2], [-x+2, 1], [y, 1], [2\,y+3, 2], [-y+x, 2], [-7\,y-2\,x, 1], [-y+x-3, 1], [-6\,y+3\,x+5, 1]]$

```
                                                                       299 msec
```

Conclusion

- Computing bounded-degree factors of lacunary polynomials
  - **Reduction** to low-degree factorization
  - Univariate: over number fields
  - Multivariate: over fields of characteristic 0
    + partial results in positive characteristic
  - Implementation: package Lacunaryx of Mathemagix

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Merci !