Factorization of lacunary polynomials

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Based on a joint work with

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Factorization: classical algorithms

Factorization of a polynomial P

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Complexity

Polynomial in the **degree** of the polynomials

The case of lacunary polynomials

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- Restriction to **some** factors only

Integral roots of integral polynomials

Gap Theorem (Cucker-Koiran-Smale'98)

Let

$$P(X) = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j}}_{Q} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j}}_{R} \in \mathbb{Z}[X]$$

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$(Y - uX - v)$ divides $P(X, Y) \iff P(X, uX + v) \equiv 0$

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with $uv \neq 0$, $\alpha_1 \leq \cdots \leq \alpha_k$. If ℓ is the smallest index s.t.

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then $P \equiv 0$ iff both $Q \equiv 0$ and $R \equiv 0$.

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then every linear factor of P divides both Q and R if $uv \neq 0$.

Example

$$
\begin{aligned}
P = X^{31}Y^6 - 2X^{30}Y^7 + X^{29}Y^8 - X^{29}Y^6 + X^{18}Y^{13} \\
- X^{16}Y^{15} + X^{17}Y^{13} + X^{16}Y^{14} + X^{10}Y^2 - X^9Y^3 \\
+ X^9Y^2 - X^5Y^6 + X^3Y^8 - 2X^3Y^7 + X^3Y^6
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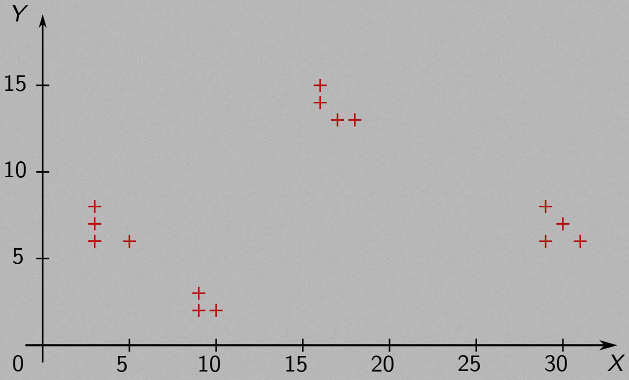

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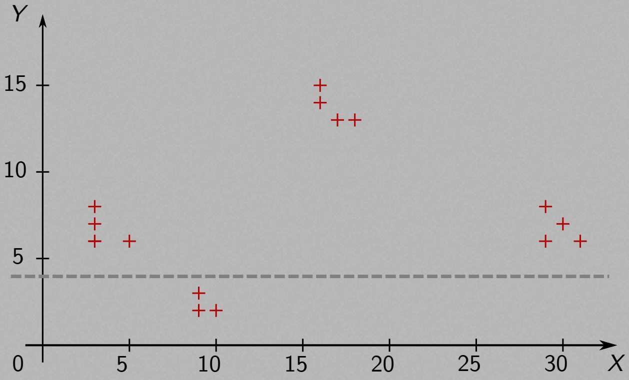

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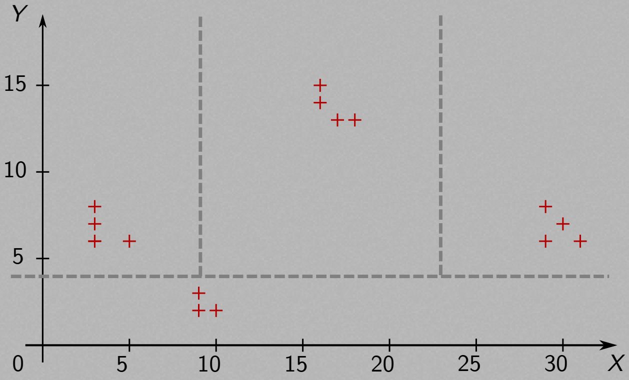

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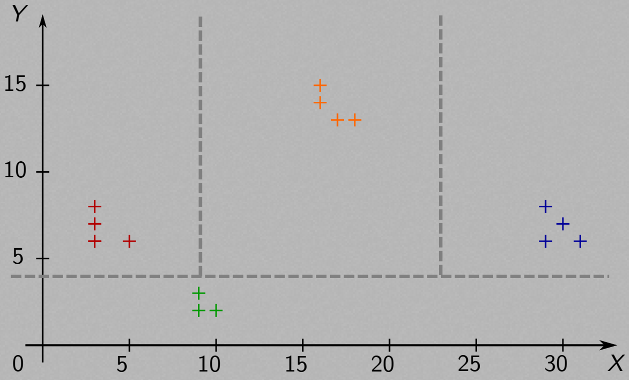

Example (2)

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Example (2)

$$-X^5Y^6 + X^3Y^8 - 2X^3Y^7 + X^3Y^6$$
$$= X^3Y^6(-X^2 + Y^2 - 2Y + 1)$$

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$\implies$ Linear factors of P: $(X - Y + 1, 1)$, $(X, 3)$, $(Y, 2)$

Valuation

$\mathbb{K}$: any field of characteristic 0

Bound on the valuation

Definition

$\mathrm{val}(P)$ = degree of the **lowest degree monomial** of $P \in \mathbb{K}[X]$

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Let

$$P = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j}(uX+v)^{\beta_j}}_{Q} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j}(uX+v)^{\beta_j}}_{R}$$

with $uv \neq 0$, $\alpha_1 \leq \cdots \leq \alpha_k$. If

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then $P \equiv 0$ iff both $Q \equiv 0$ and $R \equiv 0$.

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The Wronskian

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Let $f_1, \dots, f_\ell \in \mathbb{K}[X]$. Then

$$\mathrm{wr}(f_1, \dots, f_\ell) = \det \begin{bmatrix} f_1 & f_2 & \dots & f_\ell \\ f_1' & f_2' & \dots & f_\ell' \\ \vdots & \vdots & & \vdots \\ f_1^{(\ell-1)} & f_2^{(\ell-1)} & \dots & f_\ell^{(\ell-1)} \end{bmatrix}.$$

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Proposition (Bôcher, 1900)

$\mathrm{wr}(f_1, \dots, f_\ell) \neq 0 \iff$ the f_j's are linearly independent.

Wronskian & valuation

Lemma

$$\mathrm{val}(\mathrm{wr}(f_1,\ldots,f_\ell)) \geq \sum_{j=1}^{\ell} \mathrm{val}(f_j) - \binom{\ell}{2}$$

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$$\begin{array}{r|cccc|} & \mathrm{val}(f_1) & \mathrm{val}(f_2) & \dots & \mathrm{val}(f_\ell) \\ 0 & f_1 & f_2 & \dots & f_\ell \\ -1 & f_1' & f_2' & \dots & f_\ell' \\ \vdots & \vdots & \vdots & & \vdots \\ -(\ell-1) & f_1^{(\ell-1)} & f_2^{(\ell-1)} & \dots & f_\ell^{(\ell-1)} \end{array}$$

Upper bound for the valuation

Lemma

Let $f_j = X^{\alpha_j}(uX+v)^{\beta_j}$, $uv \neq 0$, linearly independent, and s.t. $\alpha_j, \beta_j \geq \ell$. Then

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Proof idea. Write

$$\mathrm{wr}(f_1,\dots,f_\ell) = X^{\sum_j \alpha_j - \binom{\ell}{2}}(uX+v)^{\sum_j \beta_j - \binom{\ell}{2}} \times \det(M)$$

with $\deg(M_{ij}) \leq i$.

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with $\deg(M_{ij}) \leq i$. Use $\operatorname{val}(\det M) \leq \deg(\det M) \leq \binom{\ell}{2}$.

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Let $P = \sum_{j=1}^{\ell} a_j X^{\alpha_j}(uX+v)^{\beta_j} \not\equiv 0$, with $uv \neq 0$ and $\alpha_1 \leq \cdots \leq \alpha_\ell$.

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A generalization

Theorem

Let $(\alpha_{ij}) \in \mathbb{Z}_+^{\ell\times m}$ and

$$P = \sum_{j=1}^{\ell} a_j \prod_{i=1}^{m} f_i^{\alpha_{ij}},$$

where $f_i \in \mathbb{K}[X]$, $\deg(f_i) = d_i$ and $\mathrm{val}(f_i) = \mu_i$.

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Algorithms

$\mathbb{K} = \mathbb{Q}(\alpha)$: **algebraic number field**

Algorithms

1. Polynomial Identity Testing
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- $\mathbb{K}$ is part of the input, given by φ in dense representation

Polynomial Identity Testing

Theorem

There exists a deterministic polynomial-time algorithm to test if $P = \sum_{j=1}^{k} a_j X^{\alpha_j}(uX+v)^{\beta_j}$ vanishes.

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- If $v = 0$: similar [Lenstra'99]
- If $u, v \neq 0$: $P = P_1 + \cdots + P_s$ s.t.

$$P = 0 \iff P_1 = \cdots = P_s = 0$$

where $P_t = \sum_j a_j X^{\alpha_j}(uX+v)^{\beta_j}$ with $\alpha_{\max} \leq \alpha_{\min} + \binom{k}{2}$

Polynomial Identity Testing, cont'd

$$Q(X) = \sum_{j=1}^{\ell} a_j X^{\alpha_j}(uX+v)^{\beta_j},\ \text{with } \alpha_\ell \leq \alpha_1 + \binom{\ell}{2}$$

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number of monomials, exponents $\leq$ $\text{size}(Q)^{\mathcal{O}(1)}$ $\implies$ brute force

Finding linear factors

Observation + Gap Theorem

$(Y - uX - v)$ divides $P(X, Y)$

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$$
\begin{aligned}
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$$\iff (Y - uX - v) \text{ divides each } P_t(X, Y)$$

$\rightsquigarrow$ find linear factors of low-degree polynomials

Some details

Find linear factors $(Y - uX - v)$ of $P(X, Y) = \sum_{j=1}^{k} a_j X^{\alpha_j} Y^{\beta_j}$

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   - 3.3 Apply some dense factorization algorithm to each Q_t or $\gcd(Q_1, \ldots, Q_s)$ [Kaltofen'82, ..., Lecerf'07]

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- Algebraic number field: only for Lenstra's algorithm

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Lemma

Let $P = \sum_j a_j X^{\alpha_j}(uX+v)^{\beta_j}(wX+t)^{\gamma_j} \not\equiv 0$, $uvwt \neq 0$. Then

$$\mathrm{val}(P) \leq \max_j \left(\alpha_j + 2\binom{k+1-j}{2}\right).$$

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Positive characteristic

$\mathbb{K} = \mathbb{F}_{p^s}$: **field with** p^s **elements**

Valuation

$$(1+X)^{2^n} + (1+X)^{2^{n+1}} = X^{2^n}(X+1) \mod 2$$

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Proposition

$\mathrm{wr}(f_1, \ldots, f_k) \neq 0 \iff f_j$'s linearly independent over $\mathbb{F}_{p^s}[X^p]$.

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Theorem

There exists a deterministic polynomial-time algorithm to test if $\sum_j a_j X^{\alpha_j}(uX+v)^{\beta_j} \in \mathbb{F}_{p^s}[X]$, where $p > \max_j(\alpha_j+\beta_j)$, vanishes.

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- NP-hardness: reduction from **root detection** over $\mathbb{F}_{p^s}$
  [Kipnis-Shamir'99, Bi-Cheng-Rojas'12]

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Thank you!

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