Symmetric Determinantal Representations of Polynomials

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Joint work with Erich L. Kaltofen[‡], Pascal Koiran[*†] and Natacha Portier[*†]

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Rennes – Séminaire de Calcul Formel – November 26, 2010

The problem

$$(x+3y)z = \det \begin{pmatrix} 0 & x & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & z \\ 0 & 0 & 1 & y & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Formal polynomial

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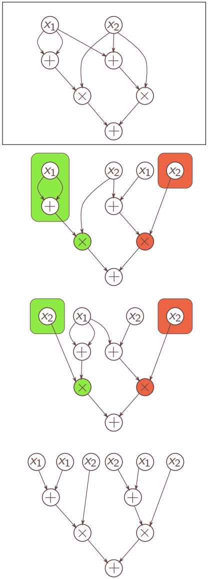
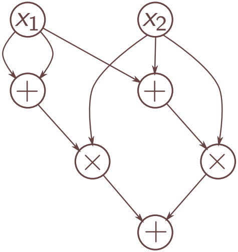
- Formal polynomial
- Smallest possible dimension of the matrix

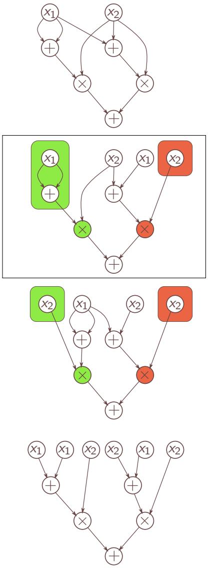
Representations of polynomials

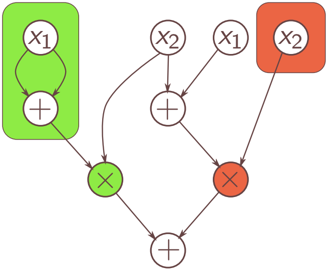




Circuit

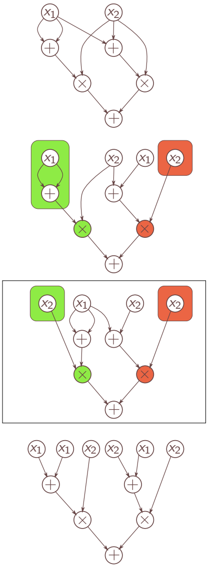
Representations of polynomials

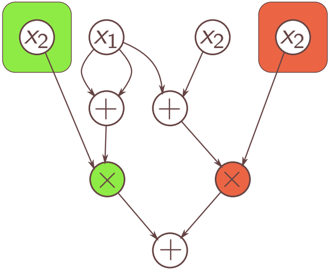




Weakly-skew circuit

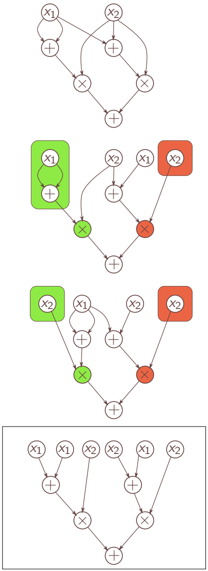
Representations of polynomials

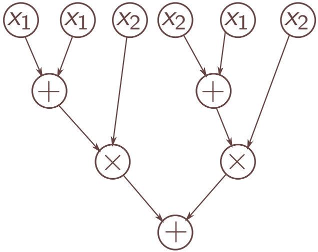




Skew circuit

Representations of polynomials





Formula

Motivation

L. G. Valiant, Completeness classes in algebra, STOC 79
$\rightsquigarrow$ Universality of the determinant

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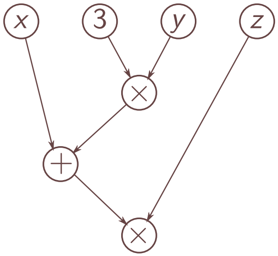
$\rightsquigarrow$ Universality of the determinant

"We conclude that for the problem of finding a subexponential formula for a polynomial when one exists, linear algebra is essentially the only technique in the sense that it is always applicable."

An example

$(x + 3y)z$



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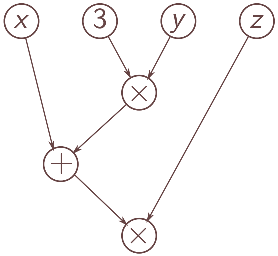

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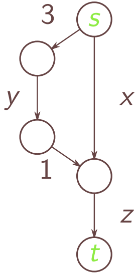
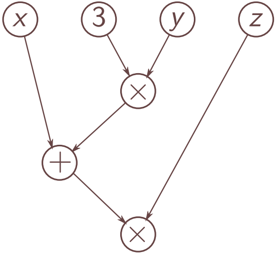
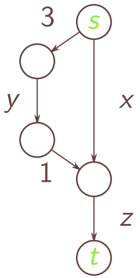




Arithmetic Branching Program

An example



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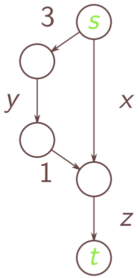

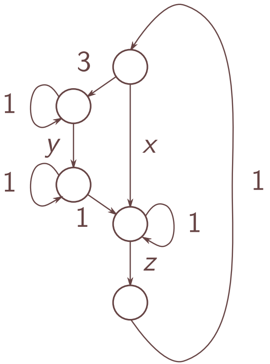

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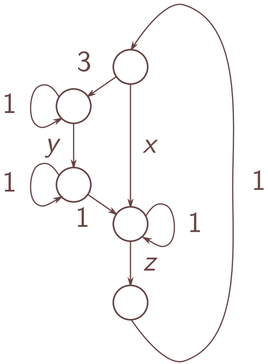

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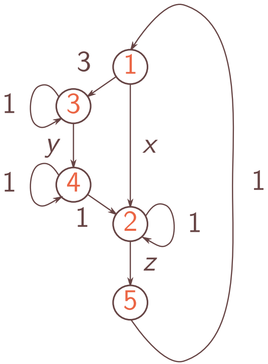

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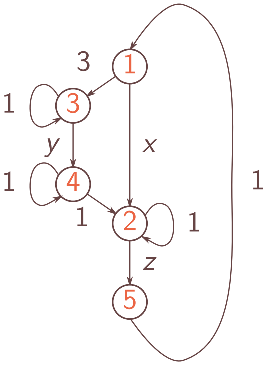

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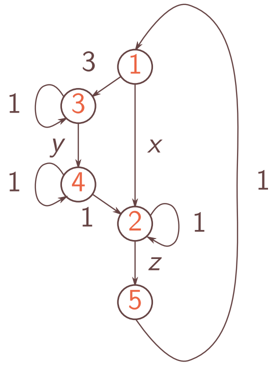

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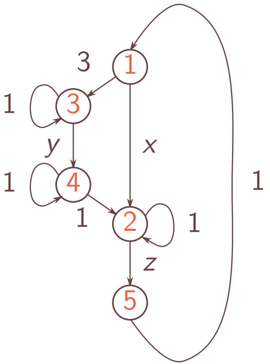

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- Up to signs, $\det A$ = sum of the weights of cycle covers in G

Outline

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- $e+1$ if there is at least one addition in the formula: H. Liu and K.W. Regan, in *Improved construction for universality of determinant and permanent* (Inf. Process. Lett., 2006)

Valiant's construction (1/3)

- Input: a formula representing a polynomial $\varphi \in \mathbb{K}[X_1, \dots, X_n]$ of size e

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- In between: a graph G of size $(e+1)$ whose adjacency matrix is A

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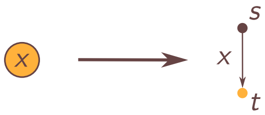

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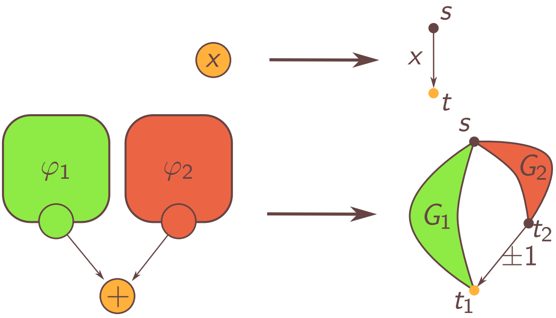

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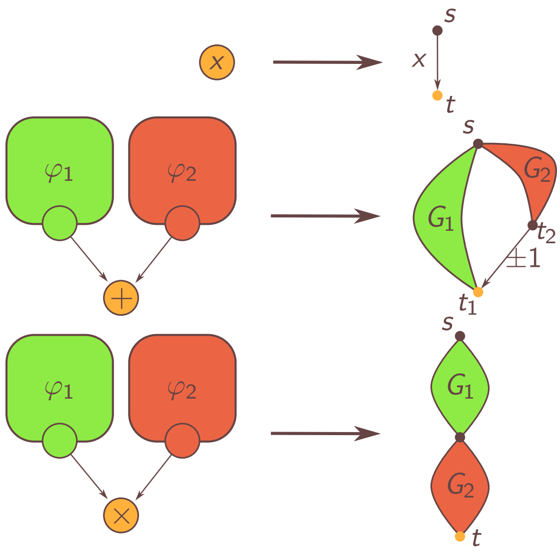

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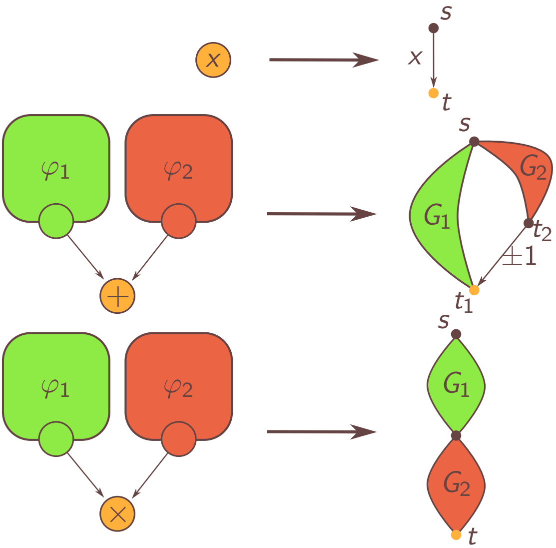

Invariant

$$\varphi = \pm \sum_{s\text{-}t\text{-paths } P} (-1)^{|P|} w(P)$$

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Theorem

For a size-e formula, this construction yields a size-$(e+1)$ graph. Let A be the adjacency matrix of G'. Then $\det(A) = \varphi$.

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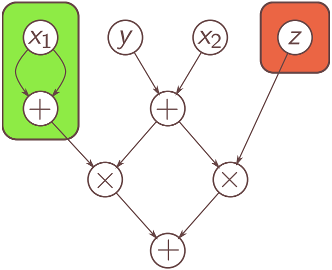

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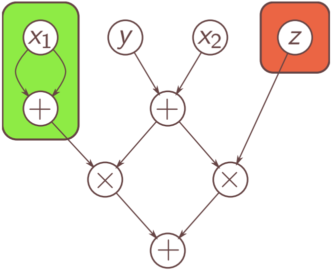

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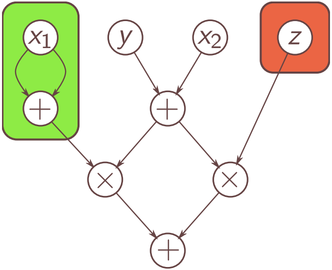
- Output: a matrix A of dimension $(e + i + 1)$ s.t. $\det A = \varphi$



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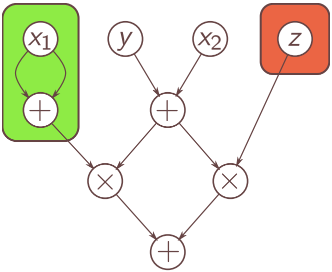

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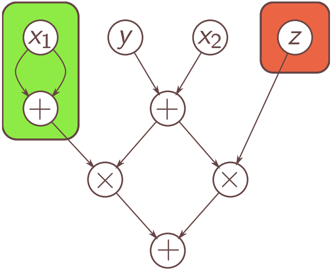

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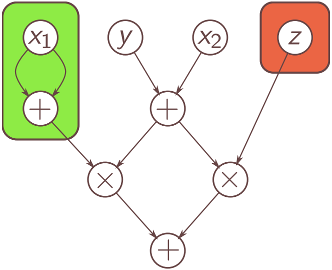
- Reusable gate: not in a closed subcircuit



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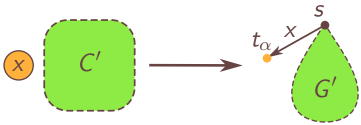

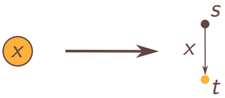

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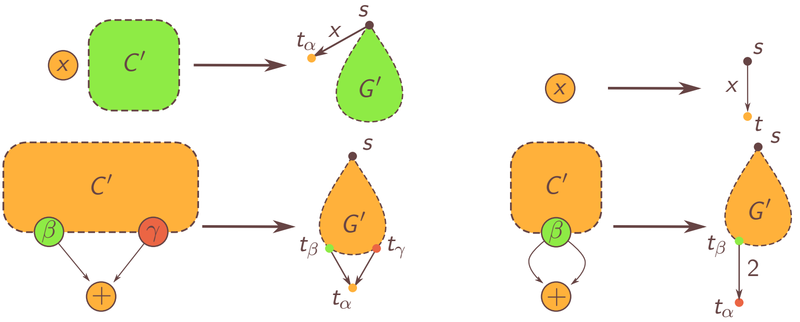

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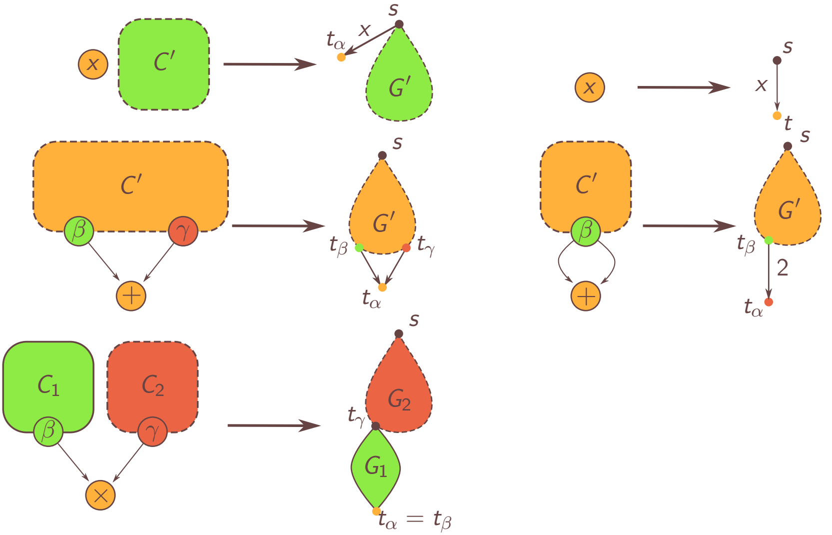

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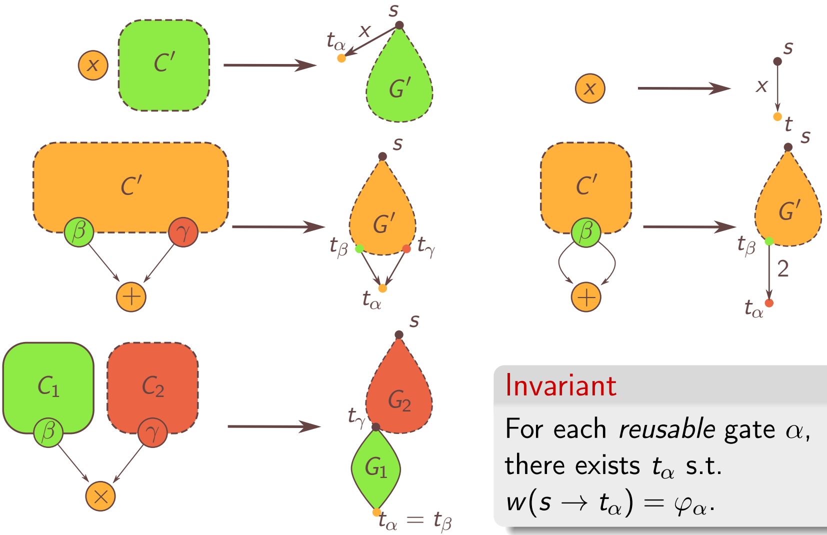

Invariant

For each *reusable* gate α, there exists t_α s.t. $w(s \to t_\alpha) = \varphi_\alpha$.

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Theorem

For a ws circuit of size e with i variable inputs representing φ, this construction yields a size-$(e+i+1)$. The determinant of its adjacency matrix equals φ.

Outline

Motivation from Convex Geometry

- Linear Matrix Expression (LME): for A_i symmetric in $\mathbb{R}^{t\times t}$

$$A_0 + x_1 A_1 + \cdots + x_n A_n$$

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- Drop condition $A_0 \succeq 0 \leadsto$ exponential size matrices
- What about polynomial size matrices?

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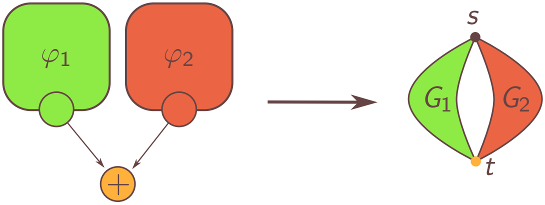

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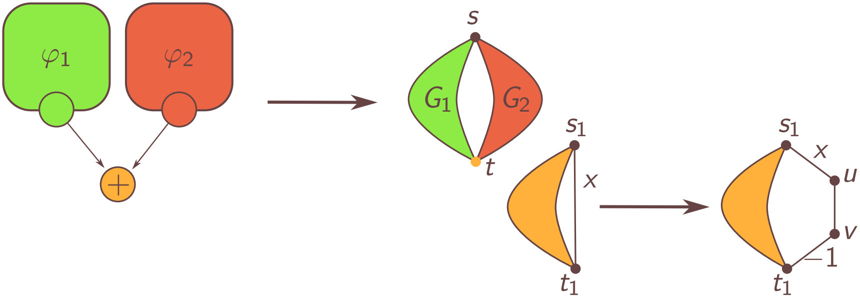

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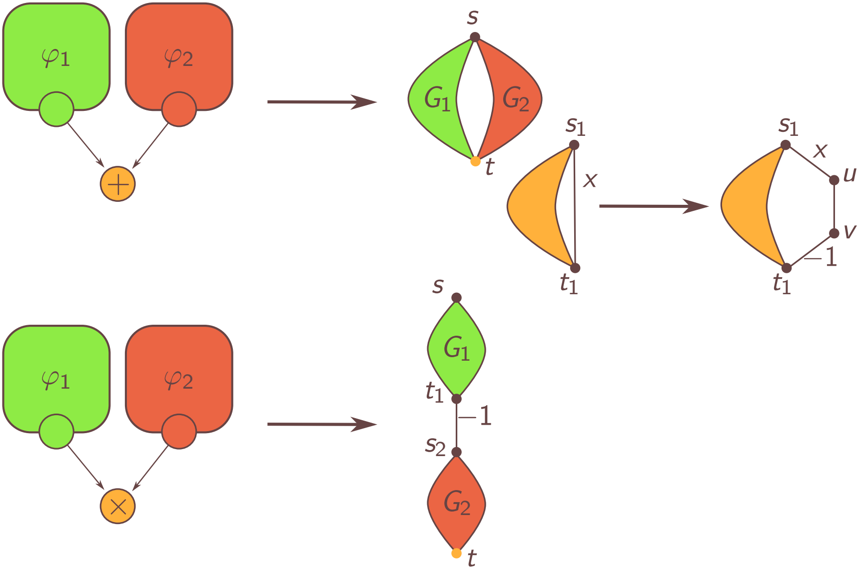
φ_1
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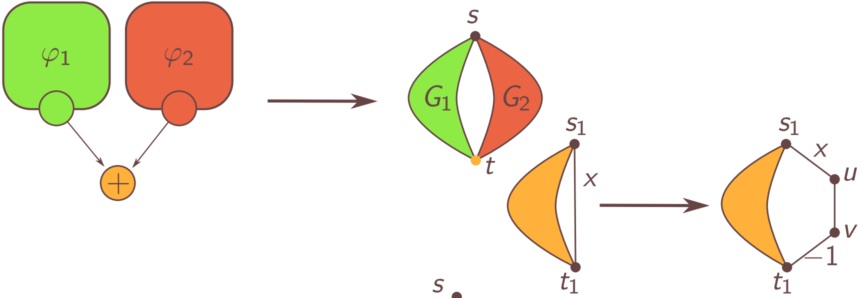
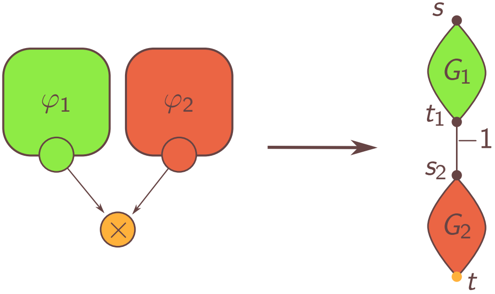
t

Algorithm





Invariants

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and. . .

Invariants for formula's construction

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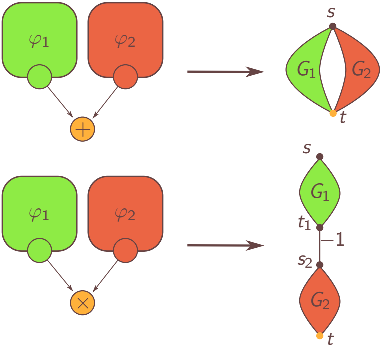

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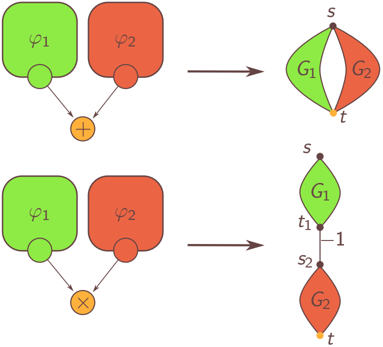

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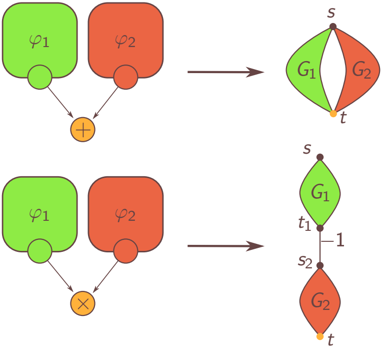

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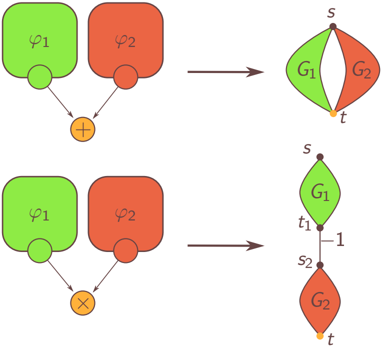

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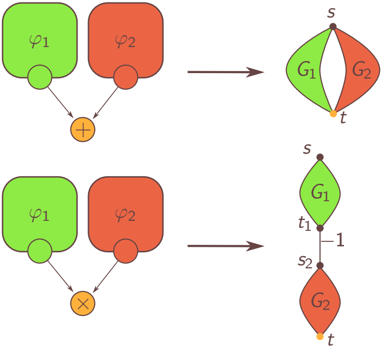

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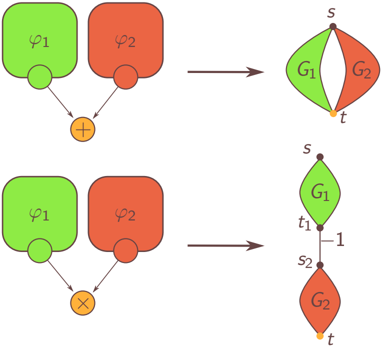

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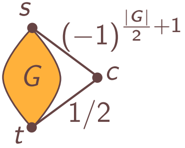

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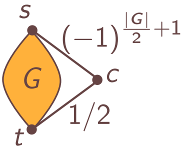

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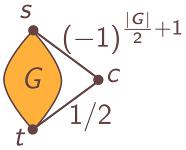

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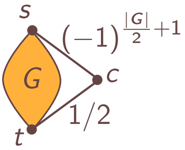

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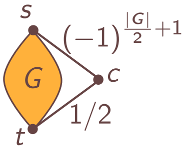

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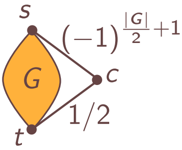

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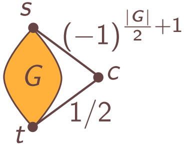

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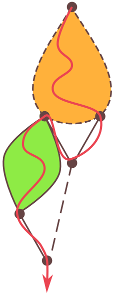
Theorem

For a formula φ of size e, this construction yields a graph of size $2e+3$. The determinant of its adjacency matrix equals φ.

Introduction

- Main difficulty:



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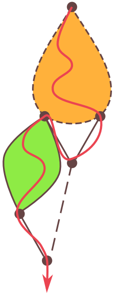
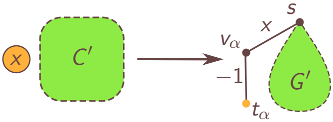


- Definition: an path P is said acceptable if $G \setminus P$ admits a cycle cover

Constructions



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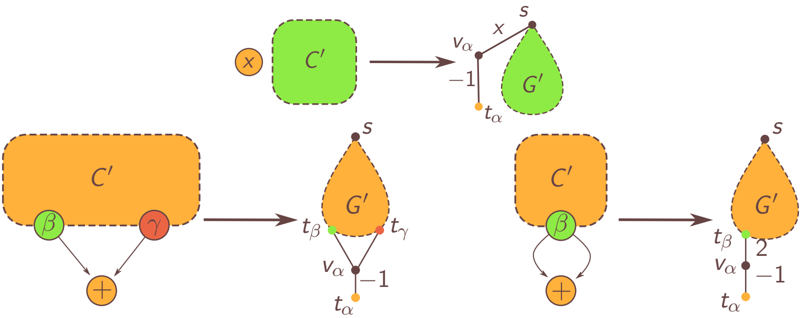

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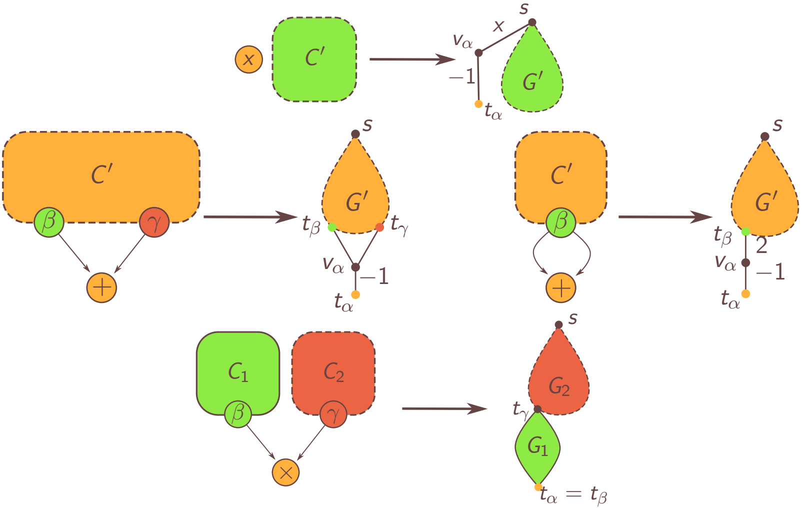

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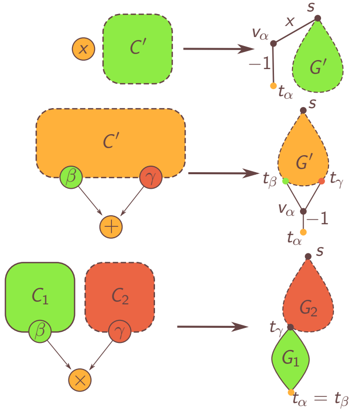

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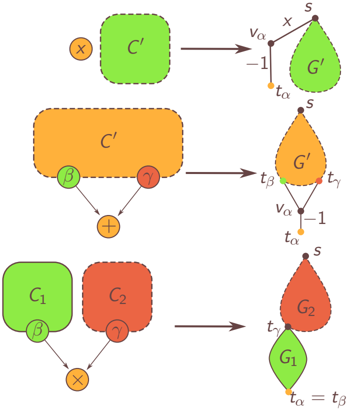

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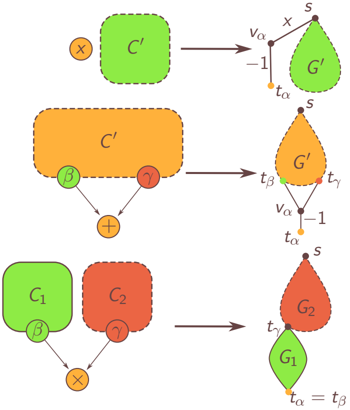

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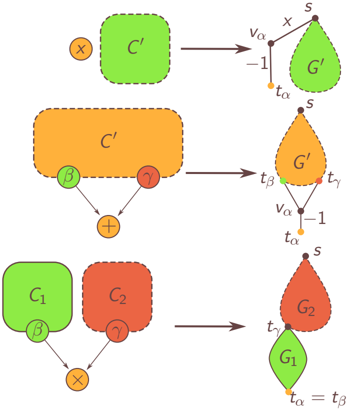

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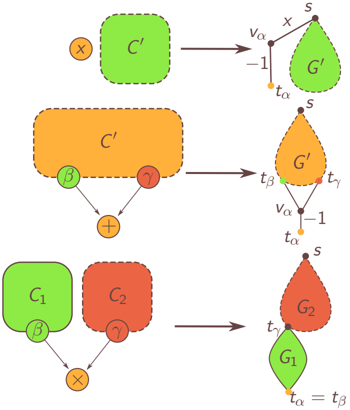

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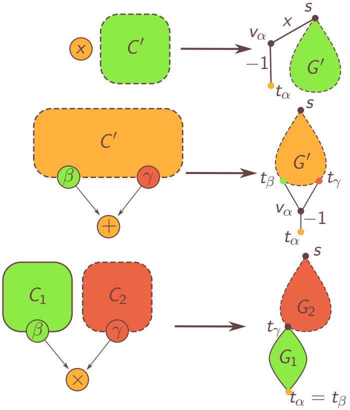

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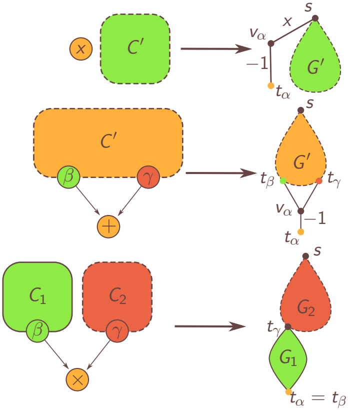

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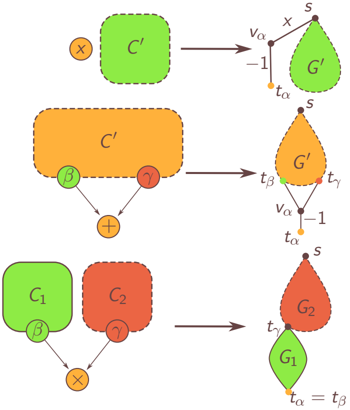

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Theorem

For a weakly skew circuit of size e, with i input variables, computing a polynomial φ, this construction yields a graph G' with $2(e+i)+1$ vertices. The adjacency matrix of G' has its determinant equal to φ.

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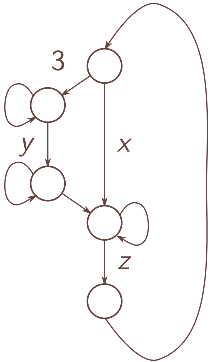

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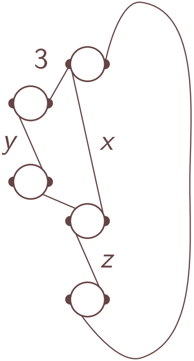

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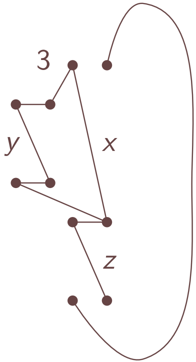

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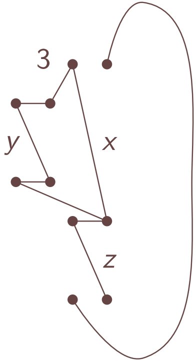

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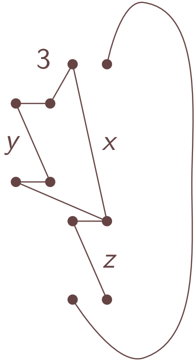

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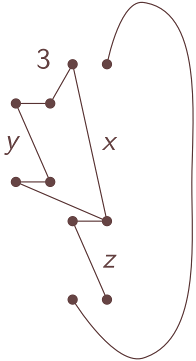

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If there exists a symmetric matrix A such that $p = \det A$, then $p \bmod \langle x^2 + \ell_x, y^2 + \ell_y, z^2 + \ell_z \rangle$ can be written as a product of degree-1 polynomials.

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A negative result

Theorem (G., Monteil, Thomassé)

If there exists a symmetric matrix A such that $p = \det A$, then $p \bmod \langle x^2 + \ell_x, y^2 + \ell_y, z^2 + \ell_z \rangle$ can be written as a product of degree-1 polynomials.

Conjecture

This is not sufficient.

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- Characterization?

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Outline

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Let p a degree-d polynomial in n variables. Then p admits a formula of size

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- Symmetric matrices in Valiant's theory?

Thank you!