Root finding over finite fields

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Given $f \in \mathbb{F}_q[X]$, compute its roots, that is $\{\alpha \in \mathbb{F}_q : f(\alpha) = 0\}$.

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- Building block for many algorithms in computer algebra: root finding over $\mathbb{Z}$, factorization, sparse interpolation, ...
- Applications in cryptography, error correcting codes, ...
- Derandomization
- Sparse interpolation: bottleneck in practice

[van der Hoeven & Lecerf, 2014]

$\mathbb{F}_q$: field with q elements, $q = p^r$ for some prime number p

- $\mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}$; $+, -, \times$ and $/$ **modulo** p
- $\mathbb{F}_q \simeq \mathbb{F}_p[\lambda]/\langle\phi\rangle$ ($\phi \in \mathbb{F}_p[\lambda]$ irreducible of degree r); $+, -, \times$ and $/$ **modulo** p **and** ϕ

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- $\mathbb{F}_3 = \{0, 1, 2\}$:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

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0	0	0	0
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;

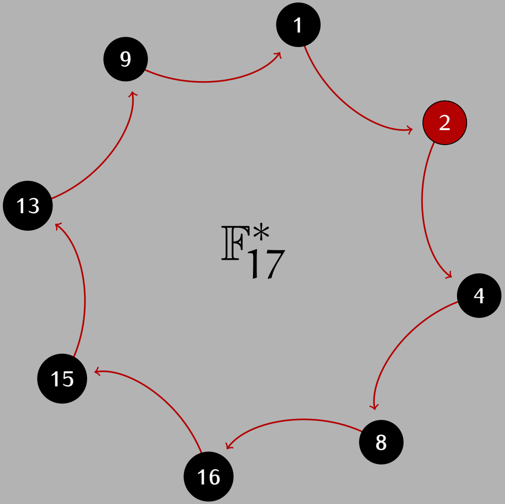
- $\mathbb{F}_4 = \mathbb{F}_2[\lambda]/\langle\lambda^2 + \lambda + 1\rangle = \{0, 1, \lambda, \lambda + 1\}$:

$+$	1	λ	$\lambda+1$
1	0	$\lambda+1$	λ
λ	$\lambda+1$	0	1
$\lambda+1$	λ	1	0

and

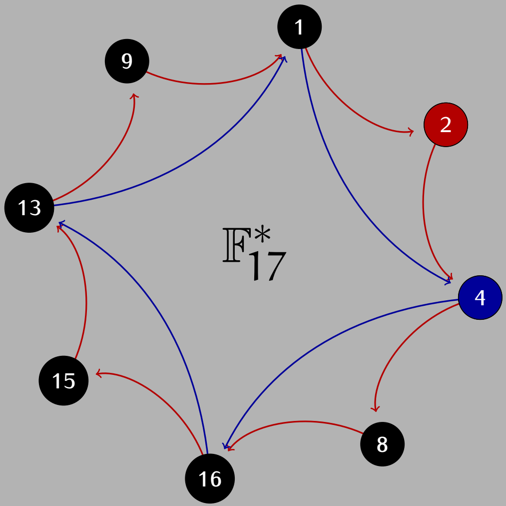
$\times$	λ	$\lambda+1$
λ	$\lambda+1$	1
$\lambda+1$	1	λ

.



2 is a primitive root of unity of order 8

Multiplicative structure



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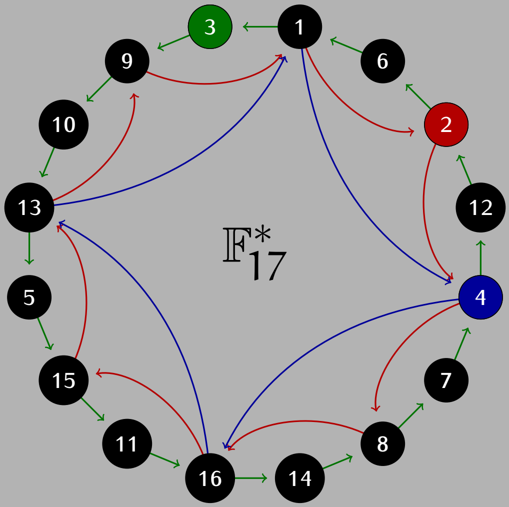

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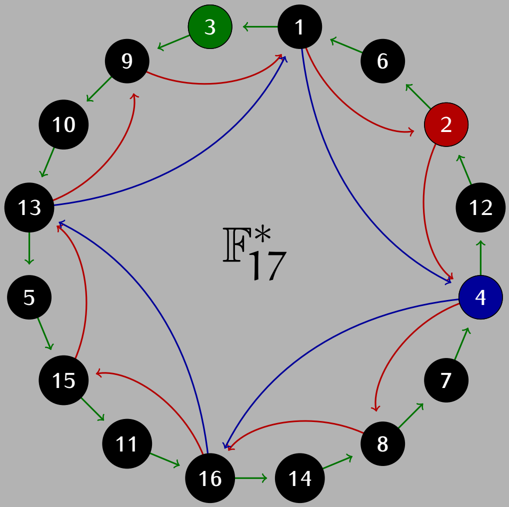

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$$\alpha^{q-1} = \zeta^{i(q-1)} = 1$$

A first algorithm

Algorithm

Input: $f \in \mathbb{F}_q[X]$
Output: The roots of f.

$Z \leftarrow \emptyset$;
for all $\alpha \in \mathbb{F}_q^*$ do
  if $f(\alpha) = 0$ then
    Add α to Z;
return Z.

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for all α ∈ F_q^* do
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Theorem

The above algorithm runs in **deterministic time** $\mathrm{poly}(q, \deg(f))$.

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⚠ The input size is $(1 + \deg(f)) \log q$: exponential time!

Algorithm

Input: $f \in \mathbb{F}_q[X]$
Output: The roots of f.

1: $Z \leftarrow \emptyset$;
2: for all $\alpha \in \mathbb{F}_q^*$ do ▷ in **random** order
3: if $f(\alpha) = 0$ then
4: Add α to Z;
5: return Z.

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The expected number of steps to discover all roots is $\frac{d}{d+1} q$.

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Obtain **fast** algorithms for polynomial root finding in finite fields.

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- **Assumption:** $f = \prod_{i=1}^{d}(X - \alpha_i)$, α_i **distinct** and **nonzero**:
  - Easy reduction to this case: $f \leftarrow \gcd(f, X^{q-1} - 1)$ ($X^{q-1} - 1 = \prod_{\alpha \in \mathbb{F}_q^*} X - \alpha$).

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- **Extra input:** A primitive element ζ, or a primitive root of unity ξ of order χ.
- **Smooth cardinality:**
  - $q = \rho\pi_1 \cdots \pi_m + 1$, where ρ, π_1, ..., π_m are *small*;
  - Practical purpose: $q = M2^m + 1$ is a *FFT prime*.

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Good and bad news

The expected number of calls is $2d$.
The complexity of step 3 is $\tilde{O}(q)$.

Cantor-Zassenhaus' method

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- $\gcd\in\{1,f\} \iff \begin{cases}\forall i, (\alpha_i+s)^{\frac{q-1}{2}} = 1 & \text{or}\\ \forall i, (\alpha_i+s)^{\frac{q-1}{2}} = -1;\end{cases}$
- $(\alpha_1+s)^{\frac{q-1}{2}} = -(\alpha_2+s)^{\frac{q-1}{2}} \neq 0 \iff \left(\frac{\alpha_1+s}{\alpha_2+s}\right)^{\frac{q-1}{2}} = -1;$
- $s\mapsto \frac{\alpha_1+s}{\alpha_2+s}$ is a bijection $\mathbb{F}_q\setminus\{-\alpha_2\}\to\mathbb{F}_q\setminus\{1\}$.

Algorithm

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Output: The roots of f.

1: if $\deg(f) = 1$ then return its root;
2: Take $s \in \mathbb{F}_q$ at random;
3: $h \leftarrow (X+s)^{\frac{q-1}{2}} \mod f$; ▷ repeated squaring
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Theorem

The above algorithm runs in expected time $\tilde{O}(d \log^2 q)$.

Modified Cantor-Zassenhaus' algorithm

Let $q = \chi\rho + 1$. Then $X^{q-1} - 1 = \prod_{i=0}^{\chi-1} (X^\rho - \xi^i)$, where $\xi^\chi = 1$.

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1: if $\deg(f) \leqslant 1$ then return its root;
2: Take $s \in \mathbb{F}_q$ at random;
3: $h \leftarrow (X+s)^\rho \mod f$; $g_0 \leftarrow f$;
4: for $i = 1$ to $\chi - 1$ do $g_i \leftarrow \gcd(g_0, h - \xi^i)$; $g_0 \leftarrow g_0/g_i$;
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If $\chi \ll \log q/\log d$, the **speed-up is approximately** $\log_2 \chi$.

Definition

The **Graeffe transform** of $g \in \mathbb{F}_q[X]$ is the unique polynomial $h \in \mathbb{F}_q[X]$ such that

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The **generalized Graeffe transform** of $g \in \mathbb{F}_q[X]$ of order π is

$$G_\pi(g)(X) = (-1)^{\pi \deg g} \operatorname{res}_z(g(z), z^\pi - x).$$

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Note. $G_{\pi_1 \pi_2} = G_{\pi_1} \circ G_{\pi_2}$

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Remark

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Remark

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Skeleton of the algorithms

1: $h_0 \leftarrow G_\rho(f)$;
2: for $i = 1$ to m do
3: $\quad h_i \leftarrow G_{\pi_i}(h_{i-1})$; $\triangleright\ h_i \leftarrow G_{\rho\pi_1\cdots\pi_i}(f)$
4: $Z_m \leftarrow \{1\}$, unique root of $h_m = G_{q-1}(f)$;
5: for $i = m-1$ down to 0 do
6: $\quad Z_i \leftarrow$ roots of h_i from Z_{i+1};
7: return the roots of f, computed from Z_0.

Lemma

Let π divide $q-1$, and ξ a primitive root of unity of order π. Then

$$G_\pi(g)(X^\pi) = g(X)g(\xi X)\cdots g(\xi^{\pi-1}X).$$

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Theorem

Given $g \in \mathbb{F}_q[X]$ and a primitive root of unity ξ of order π, $G_\pi(g)$ can be computed in $\tilde{O}(\pi d \log q)$ operations.

Improved Graeffe transform computation

Theorem

Let $g \in \mathbb{F}_q[X]$ of degree d. For all $\delta > 0$ such that $d^{1+\delta} \leqslant q-1$, $G_\pi(g)$ can be computed in time $(d \log q)^{1+\delta} + \tilde{O}(d \log q \log \pi)$.

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Based on:

Theorem [Kedlaya-Umans'11]

Let f, g, $h \in \mathbb{F}_q[X]$ of degree d. For all $\delta > 0$, $(f \circ g \mod h)$ can be computed in time $d^{1+\delta}\tilde{O}(\log q)$.

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Corollary

Let $g \in \mathbb{F}_q[X]$ and $q = \rho\pi_1 \cdots \pi_m + 1$. For all δ, $G_\rho(g)$, $G_{\rho\pi_1}(g)$, …, $G_{\rho\pi_1\cdots\pi_{m-1}}(g)$ can be computed in time $(d \log^2 q)^{1+\delta}$.

Let $q = \rho\pi_1 \cdots \pi_m + 1 = \rho\chi + 1$ and $g = G_\rho(f) = \prod_i(\alpha_i - X)$

$$\prod_{i=1}^{r}(\alpha_i - X) \xrightarrow{G_\pi} \prod_{i=1}^{r}(\alpha_i^\pi - X)$$

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ξ: primitive root of unity of order χ

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$$\begin{aligned} & \forall i, (\xi^{e_i})^\pi = \xi^{f_i} \\ \Longleftrightarrow\quad & \forall i, \pi e_i = f_i \mod \chi \\ \Longleftrightarrow\quad & \forall i, e_i \in \left\{\frac{f_i + j\chi}{\pi} : 0 \leqslant j \leqslant \pi - 1\right\} \end{aligned}$$

Algorithm

Input: $f \in \mathbb{F}_q[X]$ with $\deg(f)$ distinct roots in $\mathbb{F}_q^*$;
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1: $h_i \leftarrow$ Graeffe transform of f of order $\rho\pi_1 \cdots \pi_i$;
2: $E \leftarrow [0]$; ▷ ξ-log of the root of h_m
3: for $i = m$ down to 1 do
4: $\quad E \leftarrow \left[\frac{e + j\chi}{\pi_i} : e \in E, 0 \leqslant j \leqslant \pi_i - 1, h_{i-1}\left(\xi^{\frac{e+j\chi}{\pi_i}}\right) = 0\right]$;
5: return E.

A deterministic algorithm

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5: return E.

Theorem

If $\rho, \max_i \pi_i = O(\log q)$, the algorithm runs in time $\tilde{O}(d \log^3 q)$.

Lemma

Given $h = G_\pi(g)$, and $\{a_1, \dots, a_l\}$ its roots, one can compute the roots of g in time $\tilde{O}(\sqrt{\pi} d \log q) + (d \log q)^{1+\delta}$ for all $\delta > 0$.

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Theorem

Given $f \in \mathbb{F}_q[X]$ with $\deg(f)$ distinct roots in $\mathbb{F}_q^*$ and a primitive element of $\mathbb{F}_q^*$, the roots of f can be computed in time

$$\tilde{O}(\sqrt{S_1(q-1)} d \log^2 q) + (d \log^2 q)^{1+\delta}$$

where $S_1(q-1)$ is the largest factor of $q-1$.

Lemma

Given $h = G_\pi(g)$, and $\{a_1, \dots, a_l\}$ its roots, one can compute the roots of g in time $\tilde{O}(\sqrt{\pi} d \log q) + (d \log q)^{1+\delta}$ for all $\delta > 0$.

Theorem

Given $f \in \mathbb{F}_q[X]$ with $\deg(f)$ distinct roots in $\mathbb{F}_q^*$ and a primitive element of $\mathbb{F}_q^*$, the roots of f can be computed in time

$$\tilde{O}(\sqrt{S_1(q-1)} d \log^2 q) + (d \log^2 q)^{1+\delta}$$

where $S_1(q-1)$ is the largest factor of $q-1$.

- Best known bound for *smooth* q;
- If $q = M2^m - 1$, $M = O(\log q)$, complexity $\tilde{O}(d \log^2 q)$.

Definition

The **tangent Graeffe transform of order π** of $g \in \mathbb{F}_q[X]$ is

$$G_\pi(g(X+\varepsilon)) \in (\mathbb{F}_q[\varepsilon]/\langle\varepsilon^2\rangle)[X].$$

Remark. $G_\pi(g(X+\varepsilon)) = h(X) + \varepsilon\overline{h}(X)$ where $h = G_\pi(g)$.

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Lemma

A nonzero root β of h is a **simple root** iff $\overline{h}(\beta) \neq 0$. The corresponding root of g is $\alpha = \pi\beta h'(\beta)/\overline{h}(\beta)$.

Proof. $\overline{h}(\alpha^\pi) = \pi\alpha^{\pi-1}h'(\alpha^\pi)$.

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Proof. Given $\alpha_i \neq \alpha_j$,

$$\#\left\{\tau \in \mathbb{F}_q : (\tau+\alpha_i)^\rho = (\tau+\alpha_j)^\rho\right\} \leqslant \rho.$$

$\implies$ $G_\rho(f_\tau)$ has multiple roots for at most $\frac{d(d-1)}{2}\rho$ values of τ.

Algorithm

Input: $f \in \mathbb{F}_q[X]$ with $\deg(f)$ distinct roots in $\mathbb{F}_q^*$, $q = M2^m + 1$;
Extra input: ζ a primitive element of $\mathbb{F}_q^*$.

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1: Find the smallest $\chi = M2^{m-l}$ s.t. $\chi \geqslant d(d-1)$;
2: $h_0 + \varepsilon \overline{h}_0 \leftarrow f(X - \tau + \varepsilon)$ for some random $\tau \in \mathbb{F}_q$;
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8: Make a recursive call with $f / \prod_{a \in Z}(X - a)$.

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Heuristic

Let $q = \rho\chi + 1$ and $f \in \mathbb{F}_q[X]$ with $d = \deg(f)$ roots in $\mathbb{F}_q^*$. If $\chi \geqslant 4d$, $G_\rho(f(X+\tau))$ has $\geqslant d/3$ simple roots with probability at least $1/2$, for a random $\tau \in \mathbb{F}_q$.

Justification: holds for a random f rather than $f(X+\tau)$.

Algorithm

Input: $f \in \mathbb{F}_q[X]$ with $\deg(f)$ distinct roots in $\mathbb{F}_q^*$, $q = M2^m + 1$;
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1: Find the smallest $\chi = M2^{m-l}$ s.t. $\chi \geqslant 4d$;
2: $h_0 + \varepsilon\overline{h}_0 \leftarrow f(X - \tau + \varepsilon)$ for some random $\tau \in \mathbb{F}_q$;
3: Compute the Graeffe transform $h_l + \varepsilon\overline{h}_l$ of order 2^l;

4: $E \leftarrow [e : h_l(\zeta^e) = 0]$; ▷ ζ-log of roots of h_l

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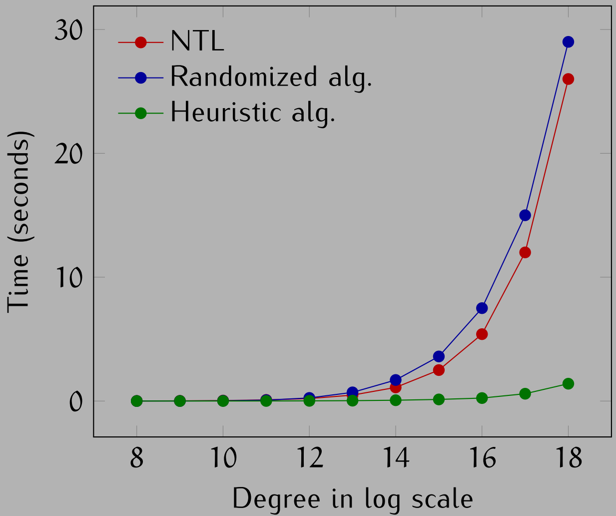
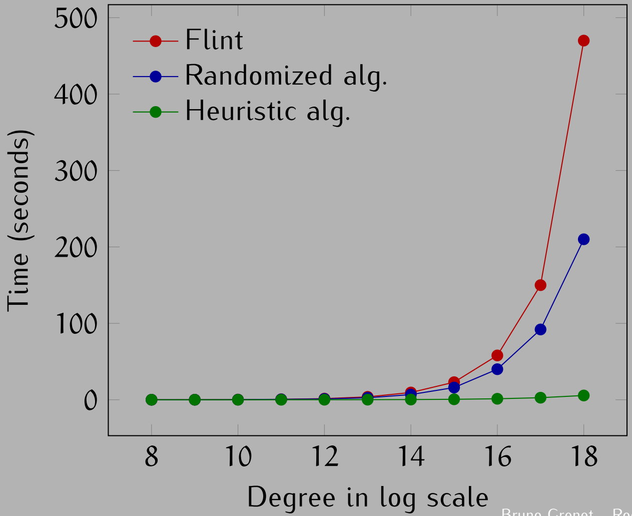
- Algorithms implemented in MATHEMAGIX (`http://mathemagix.org/`);
- Heuristic algorithm faster than FLINT and NTL by factors up to 80;
- Modification of Cantor-Zassenhaus algorithm: gain for large q only.

$q = 7 \cdot 2^{26} + 1$


NTL
Randomized alg.
Heuristic alg.
Time (seconds)
30
20
10
0
8
10
12
14
16
18
Degree in log scale

$q = 5 \cdot 2^{55} + 1$


Flint
Randomized alg.
Heuristic alg.
Time (seconds)
500
400
300
200
100
0
8
10
12
14
16
18
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Thank you!