Representations of polynomials, algorithms and lower bounds

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Representation of Univariate Polynomials

$$P(X) = X^{10} - 4\,X^8 + 8\,X^7 + 5\,X^3 + 1$$

Representations

- Dense:

$$[1, 0, -4, 8, 0, 0, 0, 5, 0, 0, 1]$$

- Sparse:

$$\Big\{(10:1), (8:-4), (7:8), (3:5), (0:1)\Big\}$$

Representation of Multivariate Polynomials

$$P(X,Y,Z) = X^2Y^3Z^5 - 4\,X^3Y^3Z^2 + 8\,X^5Z^2 + 5\,XYZ + 1$$

Representations

- Dense:

$$[1, \dots, -4, \dots, 8, \dots, 5, \dots, 1]$$

- Lacunary (supersparse):

$$\Big\{(2,3,5:1), (3,3,2:-4), (5,0,2:8), (1,1,1:5), (0:1)\Big\}$$

Representation of Multivariate Polynomials

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- Sparse:

$$\Big\{(||, |||, ||||| : 1), (|||, |||, || : -4), (|||||, , || : 8), (|, |, | : 5), (, , : 1)\Big\}$$

- Lacunary (supersparse):

$$\Big\{(2,3,5 : 1), (3,3,2 : -4), (5,0,2 : 8), (1,1,1 : 5), (0 : 1)\Big\}$$

Arithmetic Circuits

$$Q(X,Y,Z) = X^4 + 4X^3Y + 6X^2Y^2 + 4XY^3 + X^2Z + 2XYZ + Y^2Z + X^2 + Y^4 + 2XY + Y^2 + Z^2 + 2Z + 1$$

Arithmetic Circuits

$$Q(X, Y, Z) = (X + Y)^4 + (Z + 1)^2 + (X + Y)^2(Z + 1)$$

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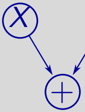

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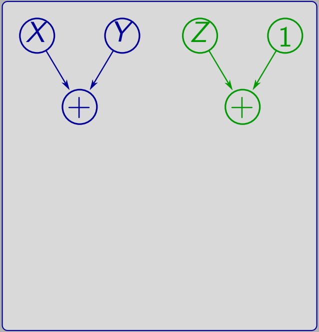

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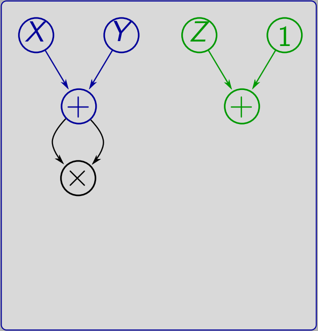

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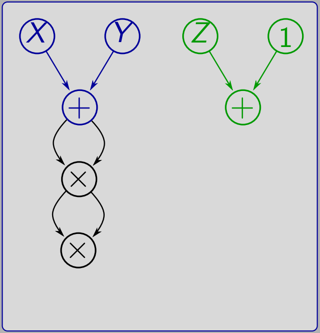

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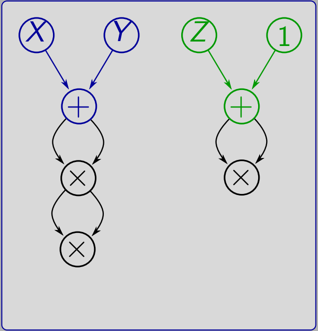

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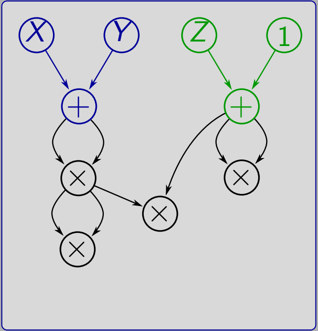

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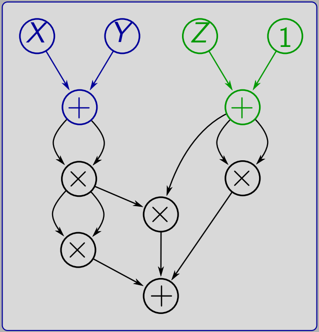
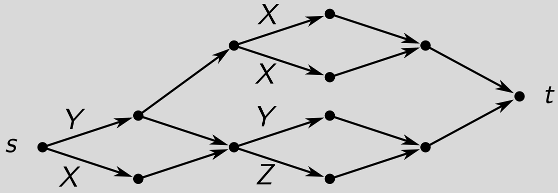


Arithmetic Branching Programs



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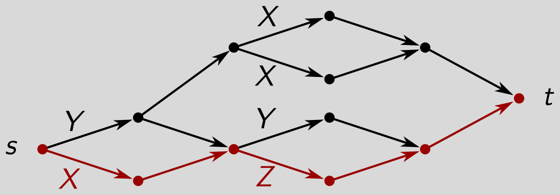
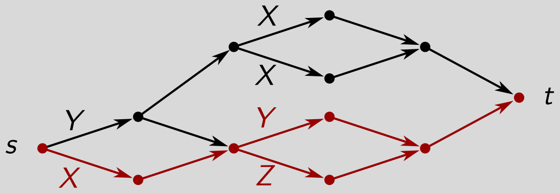


XZ

Arithmetic Branching Programs



$$X(Y+Z)$$

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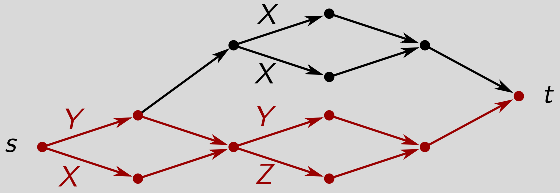
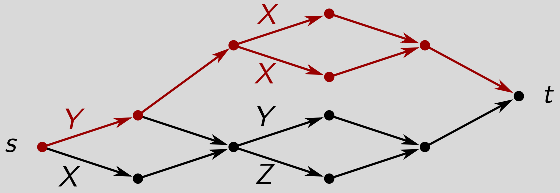


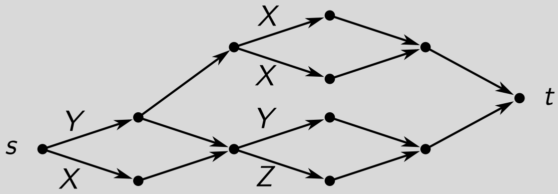
$$(X + Y)(Y + Z)$$

Arithmetic Branching Programs



$2XY$

Arithmetic Branching Programs



$$2XY + (X+Y)(Y+Z)$$

Some questions

- Links between representations

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  - Circuits
  - Branching programs
  - Determinant of matrices

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  - Factorization — lacunary
  - Polynomial Identity Testing — circuit

Outline

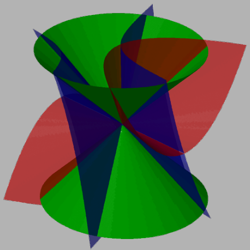
1. Resolution of polynomial systems

2. Determinantal Representations of Polynomials

3. Factorization of lacunary polynomials

1. Resolution of polynomial systems

Is there a (nonzero) solution?



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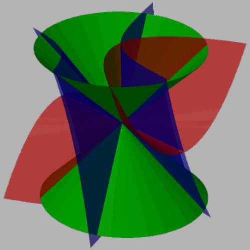

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Input: System of polynomials $f = (f_1, f_2, f_3)$, $f_j \in \mathbb{Z}[X, Y, Z]$, **homogeneous**

Question: Is there a point $a = (a_1, a_2, a_3) \in \mathbb{C}^3$, **nonzero**, s.t. $f_1(a) = f_2(a) = f_3(a) = 0$?

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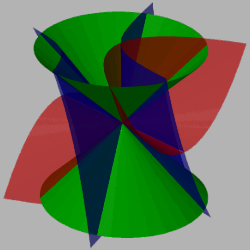

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More on the homogeneous case

Input: $f_1, \ldots, f_s \in \mathbb{K}[X_0, \ldots, X_n]$, homogeneous

Question: Is there a nonzero $a \in \bar{\mathbb{K}}^{n+1}$ s.t. $f(a) = 0$?

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  $\rightsquigarrow$ Trivial? Easy? Hard?

Definitions

PolSys($\mathbb{K}$)

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Resultant($\mathbb{K}$)

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Question: Is there a nonzero $a \in \bar{\mathbb{K}}^{n+1}$ s.t. $f(a) = 0$?

Upper bounds

Proposition (Koiran'96)

Under the Generalized Riemann Hypothesis, PolSys($\mathbb{Z}$) $\in$ AM.

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Under the Generalized Riemann Hypothesis, $\textsc{PolSys}(\mathbb{Z}) \in \mathsf{AM}$.

Corollary

Under GRH, $\textsc{HomPolSys}(\mathbb{Z})$ and $\textsc{Resultant}(\mathbb{Z})$ belong to AM.

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Positive characteristics

If p is prime, $(\textsc{Hom})\textsc{PolSys}(\mathbb{F}_p)$ & $\textsc{Resultant}(\mathbb{F}_p)$ are in PSPACE.

Known lower bounds

Notation: $\mathbb{F}_0 = \mathbb{Q}$

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For $p = 0$ or prime, PolSys($\mathbb{F}_p$) & HomPolSys($\mathbb{F}_p$) are **NP-hard**.

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$\mathbb{F}_p$	NP-hard	NP-hard	**Open**

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$\mathbb{F}_p$	NP-hard	NP-hard	**Open**

- What happens for Resultant($\mathbb{F}_p$), $p > 0$?

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Theorem (G.-Koiran-Portier'10-12)

Let p be a prime number.

- Resultant($\mathbb{F}_p$) is NP-hard for **sparse** polynomials.
- Resultant($\mathbb{F}_q$) is NP-hard for **dense** polynomials for some $q = p^s$.

Proof idea

$f(X)$: s degree-2 homogeneous polynomials in $\mathbb{F}_p[X_0, \ldots, X_n]$

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From $f(X)$ to $g(X, Y)$

$$g(X, Y) = \begin{pmatrix} f_1(X) & \\ \vdots & \text{(unchanged)} \\ f_n(X) & \\ & \\ & \\ & \end{pmatrix}$$

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- $f(a) = 0 \implies g(a,0) = 0$

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- $f(a) = 0 \implies g(a,0) = 0$
- Find λ such that $(g(a,b) = 0 \implies b = 0)$

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$f(X)$: s degree-2 homogeneous polynomials in $\mathbb{F}_p[X_0, \dots, X_n]$

From $f(X)$ to $g(X, Y)$

$$g(X,Y) = \begin{pmatrix} f_1(X) & & \\ \vdots & & \\ f_n(X) & & \\ f_{n+1}(X) & & +\lambda Y_1^2 \\ f_{n+2}(X) & - Y_1^2 & +\lambda Y_2^2 \\ \vdots & & \\ f_{s-1}(X) & - Y_{s-n-2}^2 & +\lambda Y_{s-n-1}^2 \\ f_s(X) & - Y_{s-n-1}^2 & \end{pmatrix}$$

- $f(a) = 0 \implies g(a, 0) = 0$
- Find λ such that $(g(a,b) = 0 \implies b = 0 \implies f(a) = 0)$

Conclusion

- **NP-hardness results** for square homogeneous systems of polynomials over finite fields

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- Result on the **evaluation** of the resultant polynomial

Conclusion

- **NP-hardness results** for square homogeneous systems of polynomials over finite fields
- Result on the **evaluation** of the resultant polynomial

Main open problem

- Improve the PSPACE upper bound in positive characteristics…
- … or the NP lower bound.

2. Determinantal Representations of Polynomials

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

Determinantal representations

$$
2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}
$$

- Complexity of the determinant

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "P = NP?"

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

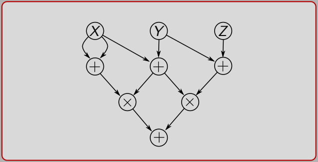
- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "P = NP?"
- Links between circuits, ABPs and the determinant

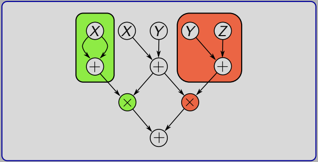
Determinantal representations

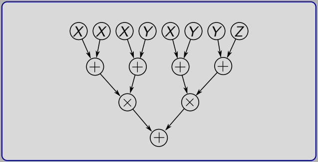
$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "P = NP?"
- Links between circuits, ABPs and the determinant
- Convex optimization

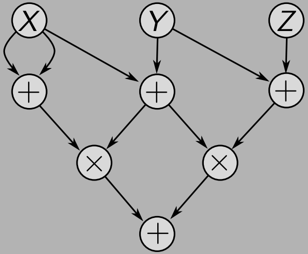
Circuits







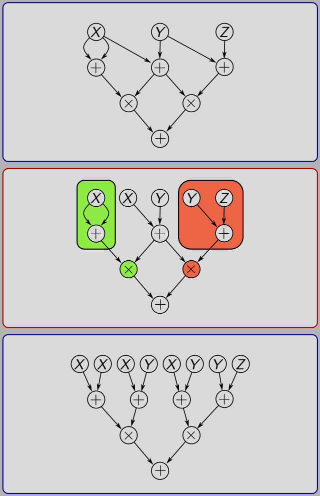
$$2X(X+Y)+(X+Y)(Y+Z)$$



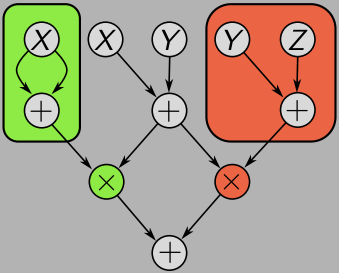
Arithmetic circuit

Size	6
Inputs	3

Circuits



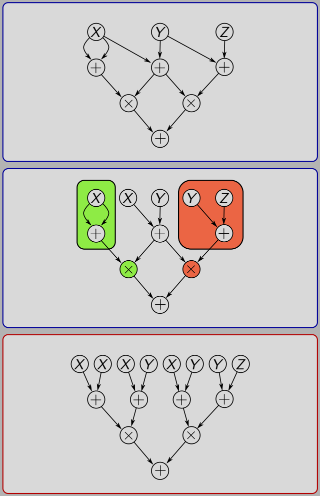
$$2X(X + Y) + (X + Y)(Y + Z)$$



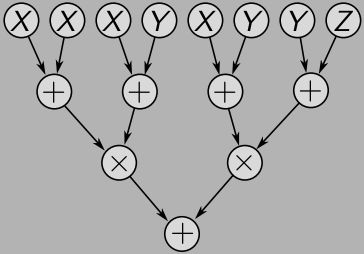
Weakly-skew circuit

Size	6
Inputs	5

Circuits



$$2X(X+Y)+(X+Y)(Y+Z)$$



Formula

Size	7
Inputs	8

Results

Proposition (Valiant'79)

Formula of **size** $s \rightsquigarrow$ Determinant of a matrix of **dimension** $(s+2)$

Results

Proposition (Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11)

Formula of size s $\rightsquigarrow$ Determinant of a matrix of dimension $(s+1)$

Results

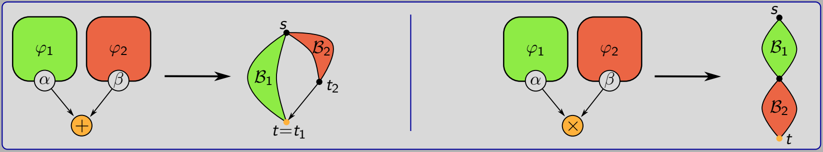
Proposition (Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11)

Formula of size $s \rightsquigarrow$ Determinant of a matrix of dimension $(s+1)$

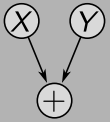
Proposition (Toda'92, Malod-Portier'08)

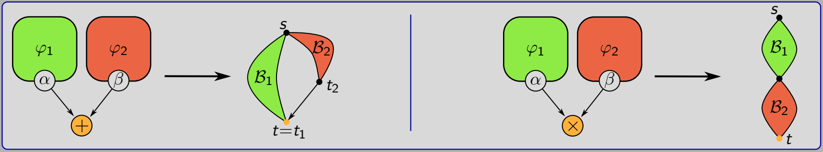
Weakly-skew circuit of size s with i inputs $\rightsquigarrow$ Determinant of a matrix of dimension $(s+i+1)$

From Formulas to Branching Programs

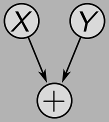


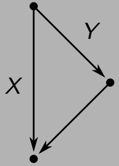
From Formulas to Branching Programs

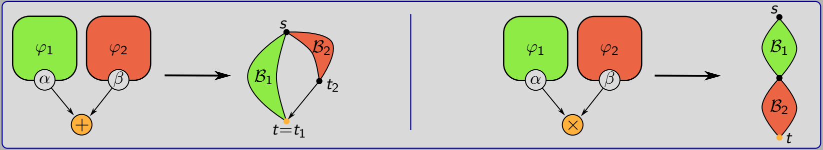




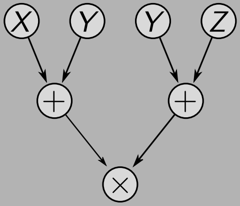
From Formulas to Branching Programs

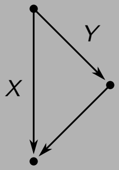


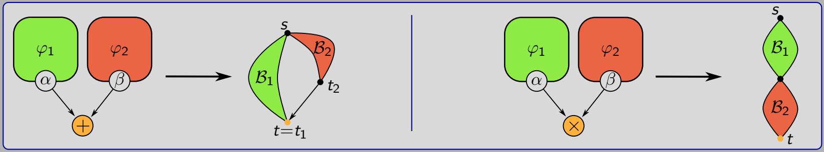




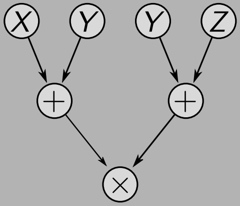
From Formulas to Branching Programs

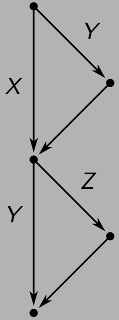


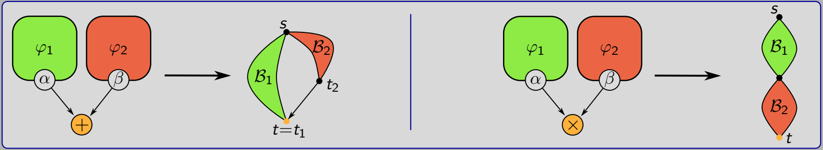




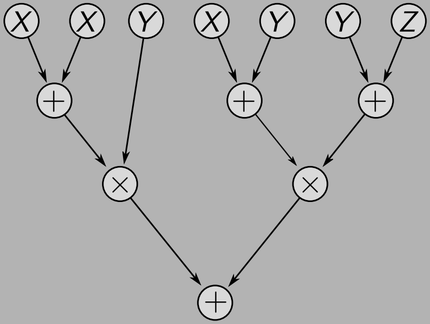
From Formulas to Branching Programs

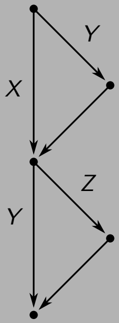


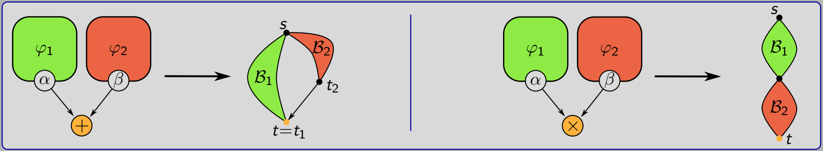




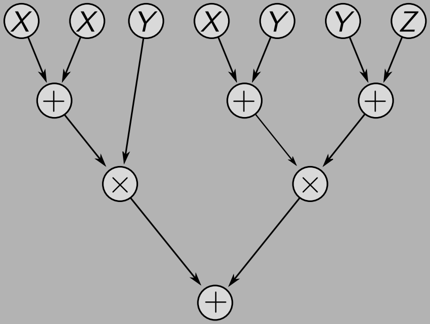
From Formulas to Branching Programs

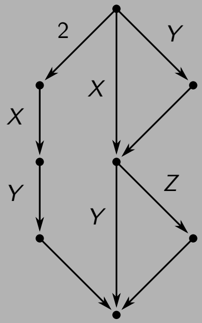


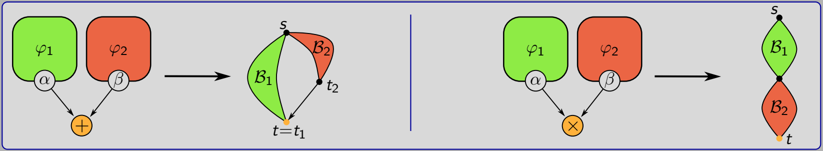




From Formulas to Branching Programs







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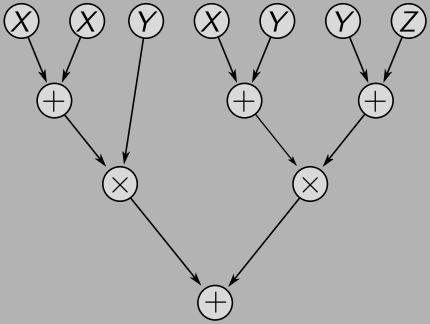
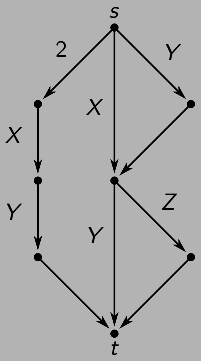
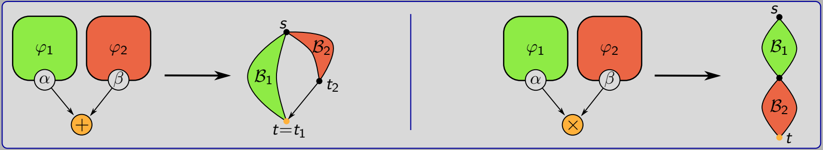
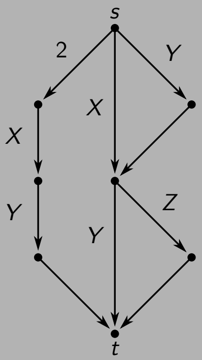


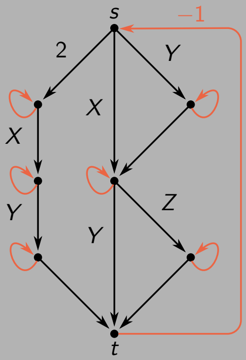




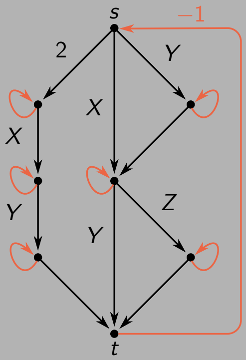
From Branching Programs to Determinants



From Branching Programs to Determinants

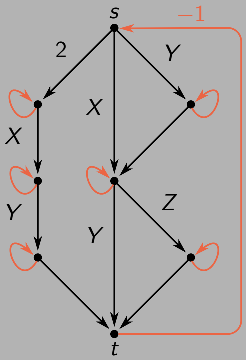


From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

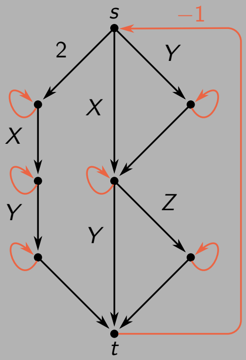
From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det M = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} M_{i,\sigma(i)}$$

From Branching Programs to Determinants

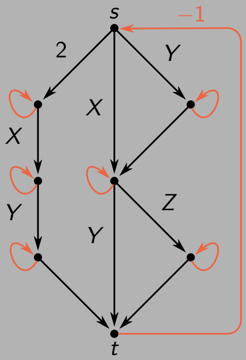


$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det M = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} M_{i,\sigma(i)}$$

- **Cycle covers $\iff$ Permutations**

From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det M = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} M_{i,\sigma(i)}$$

- **Cycle covers** $\iff$ **Permutations**
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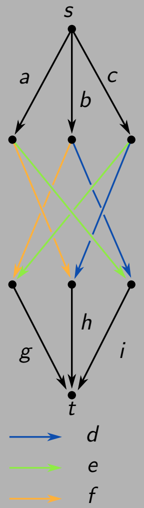

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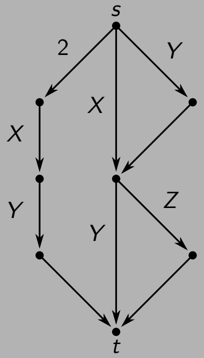
From Branching Programs to Symmetric Determinants



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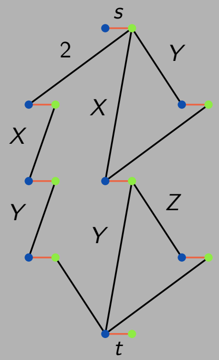

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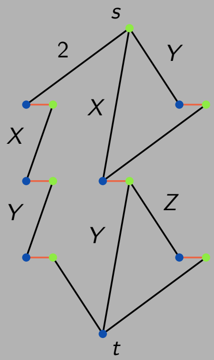

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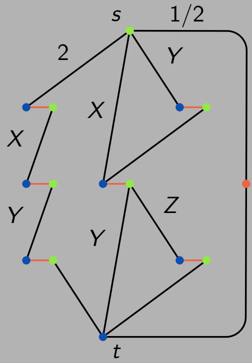

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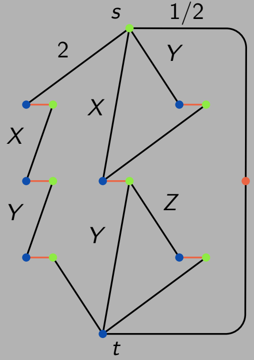

$$S = \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
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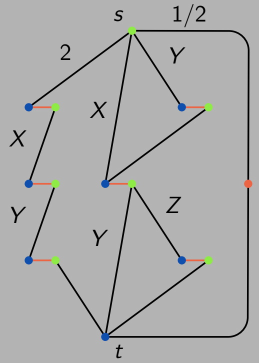

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Corollary

The **determinant of dimension n** is a projection of the **symmetric determinant of dimension $\frac{2}{3}n^3 + o(n^3)$**.

SDR in characteristic 2

$xy + yz + xz$

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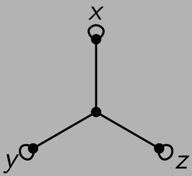

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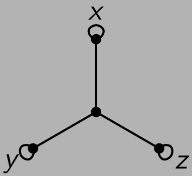

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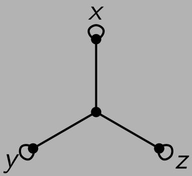

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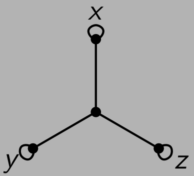

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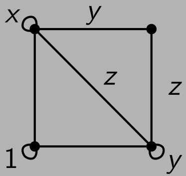

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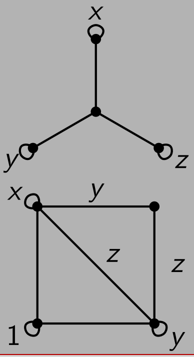

Theorem (G.–Monteil–Thomassé'12)

There are polynomials without SDR in characteristic 2, e.g. $xy+z$.

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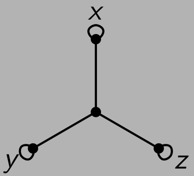

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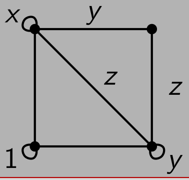

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A polynomial is said **representable** if it has an SDR.

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Determinant

$\mathfrak{S}_n$ = Permutation group of $\{1, \ldots, n\}$

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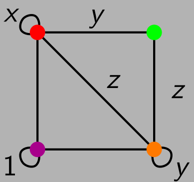

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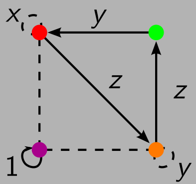

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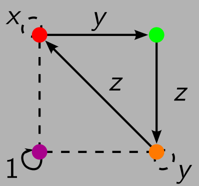

Determinant and partial matchings

Determinant in characteristic 2 of symmetric matrices

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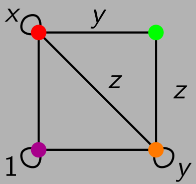

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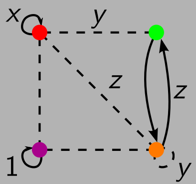

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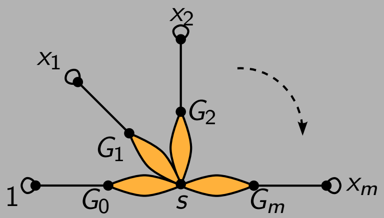

Obstructions to representability

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If P is representable, then

$$P \equiv L_1 \times \cdots \times L_k \bmod \langle x_1^2 + 1, \ldots, x_m^2 + 1 \rangle$$

where the L_i's are linear. (linear = degree-1)

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Obstructions to representability

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Such a P is said **factorizable** *modulo* $\langle x_1^2 + \ell_1^2, \ldots, x_m^2 + \ell_m^2 \rangle$.

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Theorem

Under suitable conditions, P is factorizable if and only if

$$P \equiv \mathrm{lin}(P) \times \frac{1}{\alpha_i} \frac{\partial P}{\partial x_i} \mod \langle x_1^2, \dots, x_m^2 \rangle,$$

where $\alpha_i x_i$ is a monomial of $\mathrm{lin}(P)$.

Links with coding theory?

Conjecture

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Conclusion

Same **expressiveness**:

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Main open question (Algebraic "P = NP?")

What is the **smallest** N s.t. the **permanent of dimension** n is a projection of the **determinant of dimension** N?

3. Factorization of lacunary polynomials

Introduction

Definition (reminder)

$$P(X_1,\dots,X_n)=\sum_{j=1}^{k} a_j X_1^{\alpha_{1j}}\cdots X_n^{\alpha_{nj}}$$

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Factorization: dense/sparse vs. lacunary

Factorization of a polynomial P

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$\Longrightarrow$ restriction to finding **some** factors

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$$P(X) = \sum_{j=1}^{k} a_j X^{\alpha_j} \qquad \text{size}(P) \simeq \sum_{j=1}^{k} \text{size}(a_j) + \log(\alpha_j)$$

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Theorem (H. Lenstra'99)

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Common ideas

Gap Theorem

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with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$.

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$\mathrm{gap}(P)$: function of the **algebraic height** of P.

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- Hajós' Lemma: if $\alpha_1 = \cdots = \alpha_k$, $\mathrm{val}(P) \leq \alpha_1 + (k-1)$

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The Wronskian

Definition

Let $f_1, \ldots, f_k \in \mathbb{K}[X]$. Then

$$\operatorname{wr}(f_1, \ldots, f_k) = \det \begin{bmatrix} f_1 & f_2 & \ldots & f_k \\ f_1' & f_2' & \ldots & f_k' \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \ldots & f_k^{(k-1)} \end{bmatrix}.$$

The Wronskian

Definition

Let $f_1, \dots, f_k \in \mathbb{K}[X]$. Then

$$\mathrm{wr}(f_1, \dots, f_k) = \det \begin{bmatrix} f_1 & f_2 & \dots & f_k \\ f_1' & f_2' & \dots & f_k' \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \dots & f_k^{(k-1)} \end{bmatrix}.$$

Proposition (Bôcher, 1900)

$\mathrm{wr}(f_1, \dots, f_k) \neq 0 \iff$ the f_j's are linearly independent.

Wronskian & valuation

Lemma

$$\mathrm{val}(\mathrm{wr}(f_1,\dots,f_k)) \geq \sum_{j=1}^{k} \mathrm{val}(f_j) - \binom{k}{2}$$

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Proof of the theorem. $\mathrm{wr}(P, f_2, \dots, f_k) = a_1\,\mathrm{wr}(f_1,\dots,f_k)$

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$$\sum_{j=1}^{k} \alpha_j \geq \mathrm{val}(\mathrm{wr}(f_1,\dots,f_k)) \geq \mathrm{val}(P) + \sum_{j=2}^{k} \alpha_j - \binom{k}{2}$$

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Observation + Gap Theorem

$(Y - uX - v)$ divides $P(X, Y)$

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- Find linear factors of low-degree polynomials
  $\rightsquigarrow$ [Kaltofen'82, ..., Lecerf'07]

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- $\mathbb{K} = \mathbb{Q}(\alpha)$: algebraic number field

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   - Apply some dense factorization algorithm [Kaltofen'82, …, Lecerf'07]

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- **NP-hard** under randomized reductions **otherwise**.

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Finding **multilinear factors** of **bivariate** lacunary polynomials

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Main open problem

Extend to low-degree factors of multivariate polynomials

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Thank you!