Root finding over finite fields

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- Building block for many algorithms in computer algebra: root finding over $\mathbb{Z}$, factorization, sparse interpolation, ...
- Applications in cryptography, error correcting codes, ...
- Derandomization
- Sparse interpolation: bottleneck in practice

[van der Hoeven & Lecerf, 2014]

$\mathbb{F}_q$: field with q elements, $q = p^r$ for some prime number p

- $\mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}$; $+, -, \times$ and / **modulo p**
- $\mathbb{F}_q \simeq \mathbb{F}_p[\lambda]/\langle\phi\rangle$ ($\phi \in \mathbb{F}_p[\lambda]$ irreducible of degree r); $+, -, \times$ and / **modulo p and ϕ**

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- $\mathbb{F}_3 = \{0, 1, 2\}$:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

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- $\mathbb{F}_4 = \mathbb{F}_2[\lambda]/\langle\lambda^2 + \lambda + 1\rangle = \{0, 1, \lambda, \lambda + 1\}$:

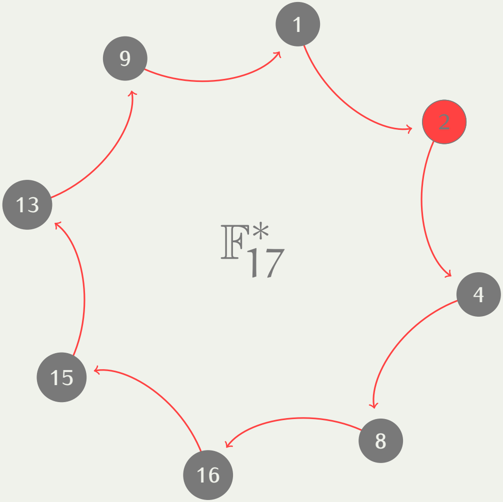
+	1	λ	$\lambda+1$
1	0	$\lambda+1$	λ
λ	$\lambda+1$	0	1
$\lambda+1$	λ	1	0

and

×	λ	$\lambda+1$
λ	$\lambda+1$	1
$\lambda+1$	1	λ

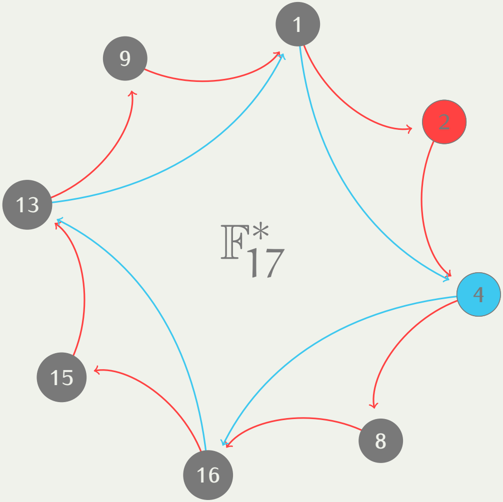
.

Multiplicative structure



2 is a primitive root of unity of order 8

Multiplicative structure



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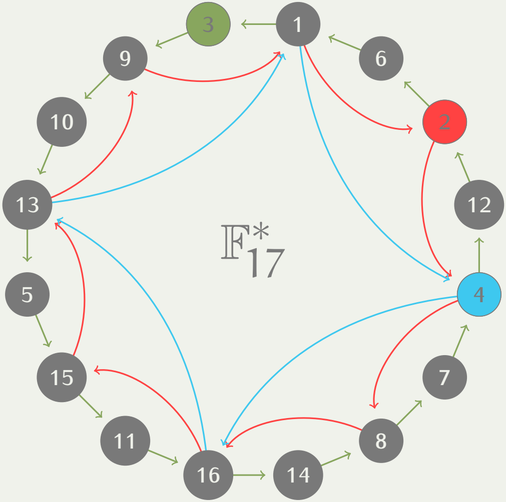

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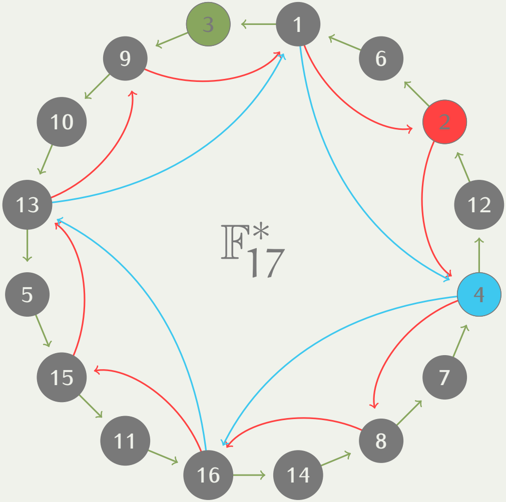

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$$\alpha^{q-1} = \zeta^{i(q-1)} = 1$$

$$X^{q-1} - 1 = \prod_{\alpha \in \mathbb{F}_q^*} (X - \alpha)$$

Theorem

The roots of $f \in \mathbb{F}_q[X]$ can be computed in **deterministic time** $\mathrm{poly}(q, d_f)$.

- **Algo:** Test each $\alpha \in \mathbb{F}_q$, sequentially.

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- Randomization: expected time $\dfrac{d_f}{d_f + 1} q$.

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Obtain **fast** algorithms for polynomial root finding in finite fields.

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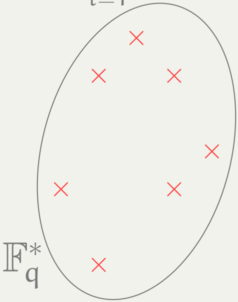
- **Extra input:** A primitive element ζ, or a primitive root of unity ξ.
- **Smooth cardinality:**
  - $q = \rho\pi_1 \cdots \pi_m + 1$, where ρ, π_1, …, π_m are *small*;
  - Practical purpose: $q = M2^m + 1$ is a *FFT prime*.

A (slow) recursive algorithm

$$f = \prod_{i=1}^{d} (X - \alpha_i)$$



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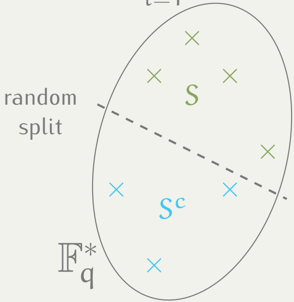

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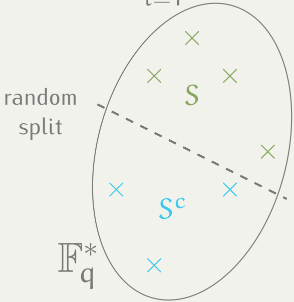
$$f = \prod_{i=1}^{d}(X - \alpha_i) \qquad u = \prod_{\alpha\in S}(X-\alpha)$$



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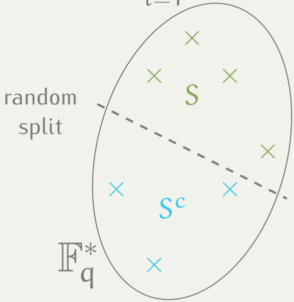

$$\xrightarrow{\gcd(f,u)} \prod_{\alpha_i \in S} (X - \alpha_i)$$

$$\xrightarrow{f/\gcd(f,u)} \prod_{\alpha_i \in S^c} (X - \alpha_i)$$

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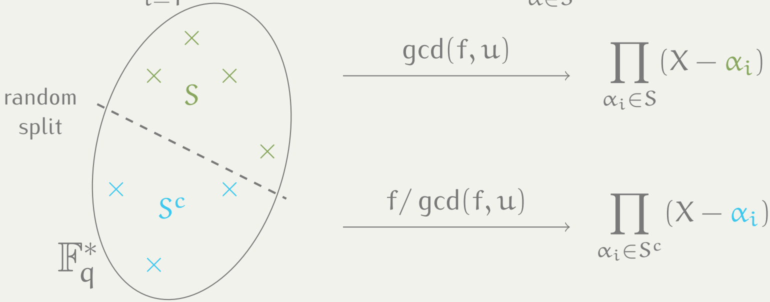

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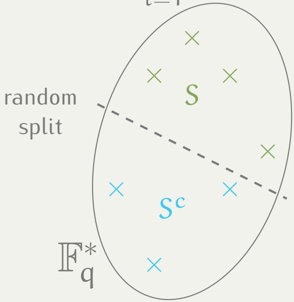

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Good and bad news

The expected number of calls is $2d$, but the complexity is $\tilde{O}(q)$.

- $\prod_{\alpha \in \mathbb{F}_q^*} (X - \alpha) = X^{q-1} - 1$

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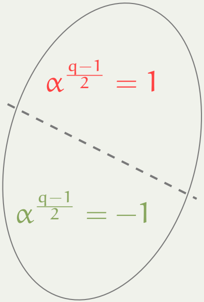

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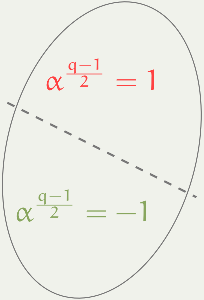

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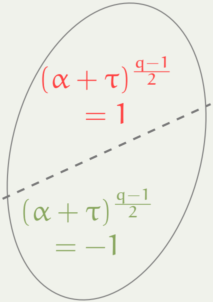

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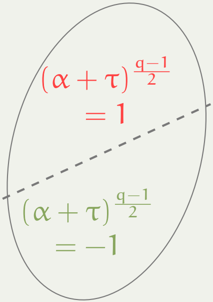

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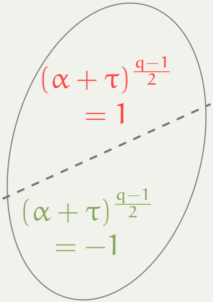

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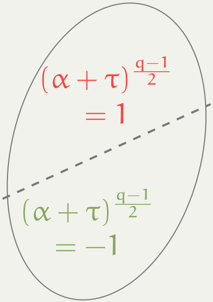

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Randomized algorithm

The roots of $f \in \mathbb{F}_q[X]$ can be computed in time $\tilde{O}(d \log^2 q)$.

Modified Cantor-Zassenhaus' algorithm
(for smooth cardinality)

Let $q = \chi\rho + 1$. Then $X^{q-1} - 1 = \prod_{i=0}^{\chi-1}(X^{\rho} - \xi^{i})$, where $\xi^{\chi} = 1$.

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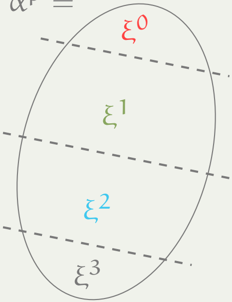

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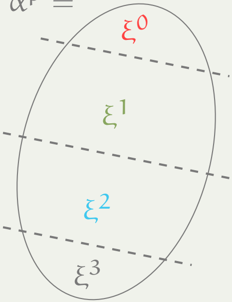

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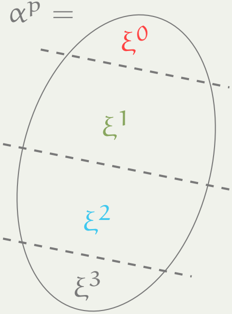

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If $\chi \ll \log q / \log d$, the **speed-up is approximately** $\log_2 \chi$.

The (generalized) Graeffe transform

Definition

The **Graeffe transform** of $g \in \mathbb{F}_q[X]$ is the unique polynomial $h \in \mathbb{F}_q[X]$ such that

$$h(X^2) = g(X)g(-X).$$

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The **generalized Graeffe transform** of $g \in \mathbb{F}_q[X]$ of order π is

$$G_\pi(g)(X) = (-1)^{\pi \deg g} \operatorname{res}_z(g(z), z^\pi - x).$$

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Note. $G_{\pi_1 \pi_2} = G_{\pi_1} \circ G_{\pi_2}$

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$$f \xrightarrow{G_\rho} h_0 \xrightarrow{G_{\pi_1}} h_1 \xrightarrow{G_{\pi_2}} \cdots \xrightarrow{G_{\pi_{m-1}}} h_{m-1} \xrightarrow{G_{\pi_m}} h_m$$

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$$h_m \longmapsto \{1\}$$

Using Graeffe transforms

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 & & & & & & & & & & \downarrow \\
Z(f) & \longleftarrow & Z_0 & \longleftarrow & Z_1 & \longleftarrow & \cdots & \longleftarrow & Z_{m-1} & \longleftarrow & \{1\}
\end{array}$$

Lemma

Let π divide $q-1$, and ξ a primitive root of unity of order π. Then

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Theorem

Given $g \in \mathbb{F}_q[X]$ and a primitive root of unity ξ of order π, $G_\pi(g)$ can be computed in $\tilde{O}(\pi d \log q)$ operations.

Improved Graeffe transform computation

Theorem

Let $g \in \mathbb{F}_q[X]$ of degree d. For all $\delta > 0$ such that $d^{1+\delta} \leqslant q - 1$, $G_\pi(g)$ can be computed in time $(d \log q)^{1+\delta} + \tilde{O}(d \log q \log \pi)$.

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Based on:

Theorem **[Kedlaya-Umans'11]**

Let $f, g, h \in \mathbb{F}_q[X]$ of degree d. For all $\delta > 0$, $(f \circ g \mod h)$ can be computed in time $d^{1+\delta}\tilde{O}(\log q)$.

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Corollary

Let $g \in \mathbb{F}_q[X]$ and $q = \rho\pi_1 \cdots \pi_m + 1$. For all δ, $G_\rho(g)$, $G_{\rho\pi_1}(g)$, $\ldots$, $G_{\rho\pi_1 \cdots \pi_{m-1}}(g)$ can be computed in time $(d \log^2 q)^{1+\delta}$.

Let $q = \rho\pi_1 \cdots \pi_m + 1 = \rho\chi + 1$ and $g = G_\rho(f) = \prod_i (\alpha_i - X)$

$$\prod_{i=1}^{r} (\alpha_i - X) \xrightarrow{G_\pi} \prod_{i=1}^{r} (\alpha_i^\pi - X)$$

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ξ: primitive root of unity of order χ

$$\prod_{i=1}^{r} (\xi^{e_i} - X) \xrightarrow{G_\pi} \prod_{i=1}^{r} (\xi^{f_i} - X)$$

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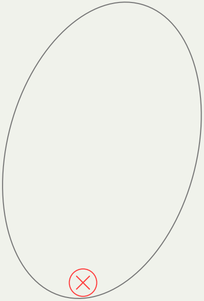
$$\prod_{i=1}^{r} (\alpha_i - X) \xrightarrow{G_\pi} \prod_{i=1}^{r} (\alpha_i^\pi - X)$$

ξ: primitive root of unity of order χ

$$\prod_{i=1}^{r} (\xi^{e_i} - X) \xrightarrow{G_\pi} \prod_{i=1}^{r} (\xi^{f_i} - X)$$

$$\begin{aligned} & \forall i, (\xi^{e_i})^\pi = \xi^{f_i} \\ \iff & \forall i, \pi e_i = f_i \mod \chi \\ \iff & \forall i, e_i \in \left\{ \frac{f_i + j\chi}{\pi} : 0 \leqslant j \leqslant \pi - 1 \right\} \end{aligned}$$

A deterministic algorithm



$Z_m = \{\xi^0\}$

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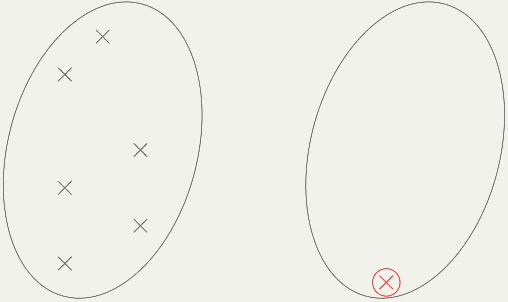

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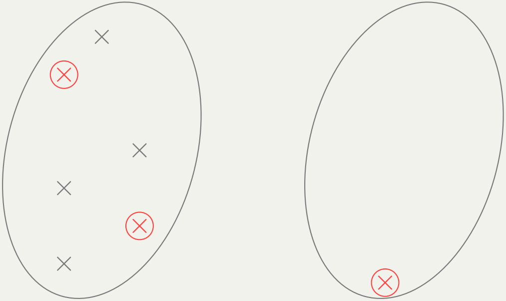

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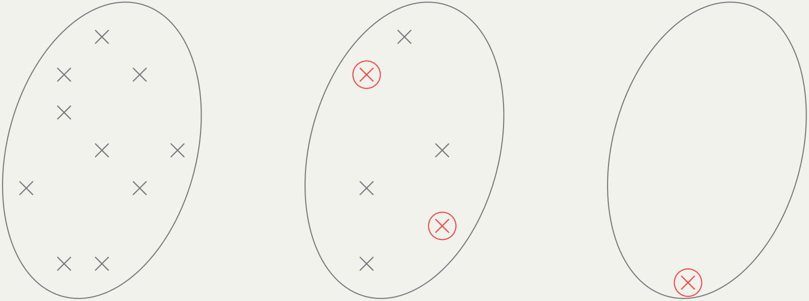

$\{\xi^{\frac{e+j\chi}{\pi_{m-1}}} : 0 \leqslant j \leqslant \pi_{m-1}\}$ $\{\xi^{\frac{j\chi}{\pi_m}} : 0 \leqslant j \leqslant \pi_m\}$ $Z_m = \{\xi^0\}$

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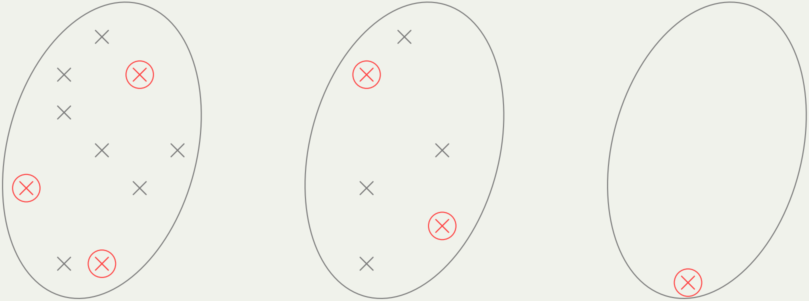

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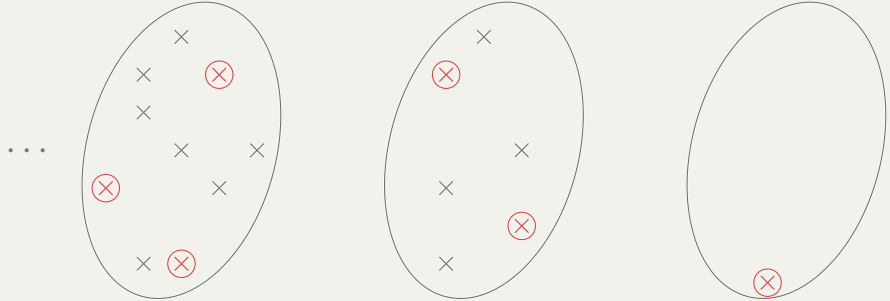

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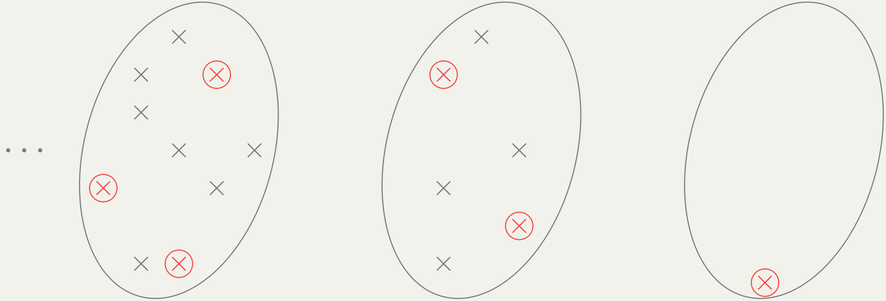

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Theorem

If $\rho, \max_i \pi_i = O(\log q)$, the algorithm runs in time $\tilde{O}(d \log^3 q)$.

Lemma

Given $h = G_\pi(g)$, and $\{a_1, \ldots, a_l\}$ its roots, one can compute the roots of g in time $\tilde{O}(\sqrt{\pi} d \log q) + (d \log q)^{1+\delta}$ for all $\delta > 0$.

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where $S_1(q-1)$ is the largest factor of $q-1$.

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where $S_1(q-1)$ is the largest factor of $q-1$.

- Best known bound for *smooth* q;
- If $q = M2^m + 1$, $M = O(\log q)$, complexity $\tilde{O}(d \log^2 q)$.

Definition

The **tangent Graeffe transform of order π** of $g \in \mathbb{F}_q[X]$ is

$$G_\pi(g(X+\varepsilon)) \in (\mathbb{F}_q[\varepsilon]/\langle\varepsilon^2\rangle)[X].$$

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Lemma

A nonzero root β of h is a **simple root** iff $\overline{h}(\beta) \neq 0$. The corresponding root of g is $\alpha = \pi\beta h'(\beta)/\overline{h}(\beta)$.

Proof. $\overline{h}(\alpha^\pi) = \pi\alpha^{\pi-1}h'(\alpha^\pi)$.

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If $q = \rho\chi + 1$ with $\chi \geqslant d(d-1)$,

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Proof. Given $\alpha_i \neq \alpha_j$,

$$\#\left\{\tau \in \mathbb{F}_q : (\tau+\alpha_i)^\rho = (\tau+\alpha_j)^\rho\right\} \leqslant \rho.$$

$\Longrightarrow$ $G_\rho(f_\tau)$ has multiple roots for at most $\frac{d(d-1)}{2}\rho$ values of τ.

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 & & & & & & & & \downarrow \\
 & & & & Z_l & \longleftarrow & \cdots & \longleftarrow & Z_m \\
 & & & & & & & & \cap \\
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- Recursive call with $f / \prod_{\alpha \in Z_0}(X - \alpha)$.

Theorem

If $q = M2^m + 1$ with $M = O(\log q)$, the randomized algorithm runs in expected time $\tilde{O}(d \log^2 q)$.

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Heuristic

Let $q = \rho\chi + 1$ and $f \in \mathbb{F}_q[X]$ with $d = \deg(f)$ roots in $\mathbb{F}_q^*$. If $\chi \geqslant 4d$, $G_\rho(f(X+\tau))$ has $\geqslant d/3$ simple roots with probability at least $1/2$, for a random $\tau \in \mathbb{F}_q$.

Justification: holds for a random f rather than $f(X+\tau)$.

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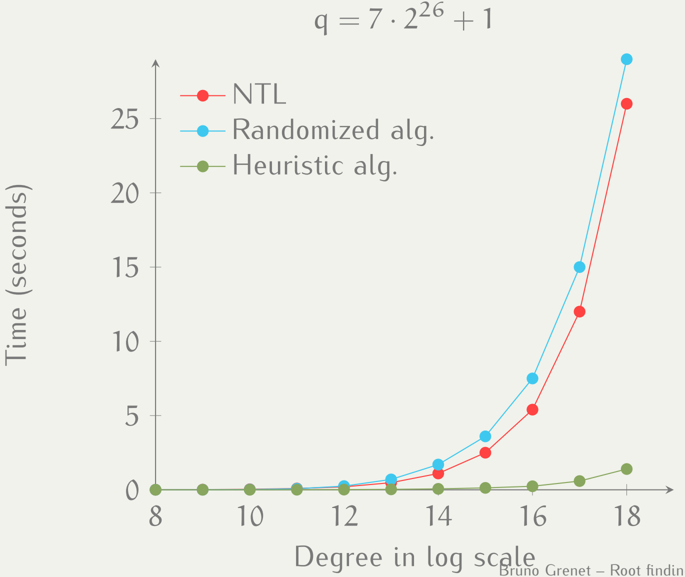
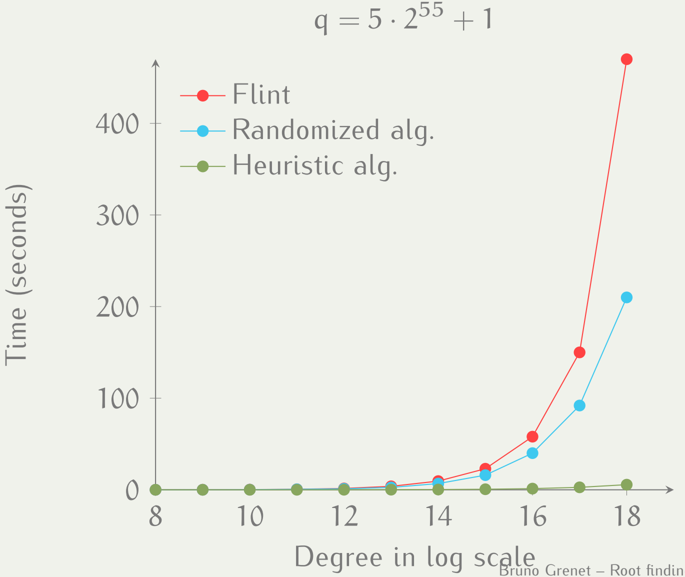
- Algorithms implemented in Mathemagix (`http://mathemagix.org/`);
- Heuristic algorithm faster than flint and ntl by factors up to 80;
- Modification of Cantor-Zassenhaus algorithm: gain for large q only.



$q = 7 \cdot 2^{26} + 1$
NTL
Randomized alg.
Heuristic alg.
Time (seconds)
0
5
10
15
20
25
8
10
12
14
16
18
Degree in log scale



$q = 5 \cdot 2^{55} + 1$
Flint
Randomized alg.
Heuristic alg.
Time (seconds)
0
100
200
300
400
8
10
12
14
16
18
Degree in log scale

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Thank you!