Complexity of the resultant

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joint work with Pascal Koiran & Natacha Portier

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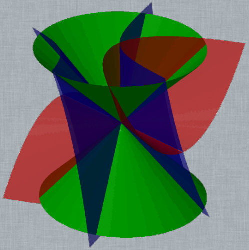
Is there a (nonzero) solution?



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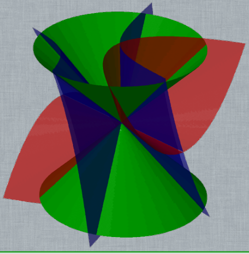

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PolSys$(\mathbb{K})$

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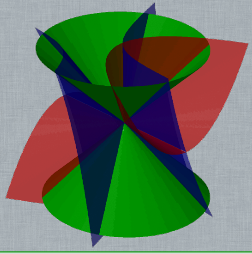

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HomPolSys($\mathbb{K}$)

Input: $f_1, \ldots, f_s \in \mathbb{K}[X_0, \ldots, X_n]$, **homogeneous**

Question: Is there a **nonzero** $\mathbf{a} \in \overline{\mathbb{K}}^{n+1}$ s.t. $f(\mathbf{a}) = 0$?

Glimpse of Elimination Theory

$$f_1, \dots, f_s \in \mathbb{K}[X_1, \dots, X_n], \qquad f_i = \sum_{|\alpha|_1 \leqslant d_i} \gamma_{i,\alpha} \mathbf{X}^\alpha$$

For which $\gamma_{i,\alpha}$ is there a root?

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There exist $R_1, \dots, R_h \in \mathbb{K}[\boldsymbol{\gamma}]$ s.t.

$$\begin{cases} R_1(\boldsymbol{\gamma}) & = & 0 \\ & \vdots & \\ R_h(\boldsymbol{\gamma}) & = & 0 \end{cases} \implies \exists \mathbf{a}, \quad \begin{cases} f_1(\mathbf{a}) & = & 0 \\ & \vdots & \\ f_s(\mathbf{a}) & = & 0 \end{cases}$$

Two Univariate Polynomials

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- Non trivial root?

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Resultant($\mathbb{K}$)

Input: $f_1, \ldots, f_{n+1} \in \mathbb{K}[X_0, \ldots, X_n]$, homogeneous

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Macaulay matrices

- $f_1, \ldots, f_{n+1} \in \mathbb{K}[X_0, \ldots, X_n]$, homogeneous, of degrees $d_1, \ldots, d_n$
- $D = \sum_i (d_i - 1)$, $\mathcal{M}_D^n = \{X_0^{\alpha_0} \cdots X_n^{\alpha_n} : \alpha_0 + \ldots + \alpha_n = D\}$

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Definition

The first Macaulay matrix is defined as follows:

- Its rows and columns are indexed by $\mathcal{M}_D^n$;
- The row indexed by $\boldsymbol{X^\alpha}$ represents

$$\frac{\boldsymbol{X^\alpha}}{X_i^{d_i}} f_i, \text{ where } i = \min\{j : d_j \leqslant \alpha_j\}.$$

Other Macaulay matrices are defined by reordering the f_i's.

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- Resultant : GCD of the determinants of n Macaulay matrices

Canny's upper bound

Theorem [Canny'87]

The resultant is computable in polynomial space.

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Proof idea.

- The resultant can be expressed as $\det(M)/\det(N)$, where M is Macaulay, and N a submatrix of M;
- An entry of M (resp. N) can be computed in polynomial time;
- The determinant can be computed in logarithmic space.

Large determinants

Theorem [G.-Koiran-Portier'10-13]

- Macaulay matrices can be represented by polynomial-size boolean circuits.
- Deciding the nullity of the determinant of a matrix represented by a boolean circuit is PSPACE-complete (over any field).

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Proof idea.

- Let $\mathcal{M}$ be a PSPACE Turing Machine;
- Let $\mathcal{G}_{\mathcal{M}}^{x}$ its *graph of configurations*:
  - initial configuration c_i,
  - accepting configuration c_a;
- $\mathcal{G}_{\mathcal{M}}^{x}$ can be represented by a boolean circuit;
- There exists a path $c_i \rightsquigarrow c_a$ in $\mathcal{G}_{\mathcal{M}}^{x}$ iff $x \in \mathcal{L}(\mathcal{M})$;
- Let $A \simeq$ adjacency matrix of $\mathcal{G}_{\mathcal{M}}^{x}$:

$$\det(A) \neq 0 \iff \exists c_i \rightsquigarrow c_a.$$

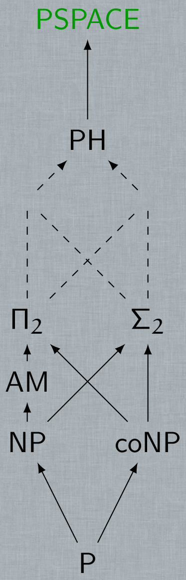
The resultant in Valiant's model of computation

Theorem

In Valiant's algebraic model of computation:

- The resultant belongs to VPSPACE, [Koiran-Perifel'07]
- Determinants of *succinctly represented* matrices is VPSPACE-complete. [Malod'11]

Upper bounds for polynomial systems



Upper bounds

- $\textsc{PolSys}(\mathbb{F}_p) \in \mathsf{PSPACE}$

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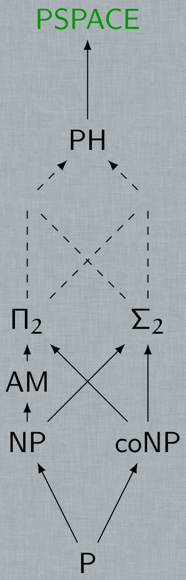

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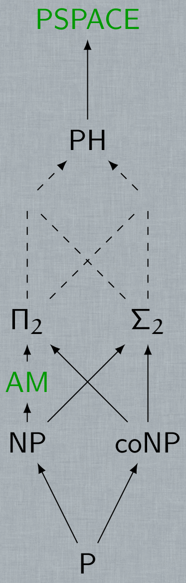
  $\implies \textsc{HomPolSys}(\mathbb{F}_p), \textsc{Resultant}(\mathbb{F}_p) \in \mathsf{PSPACE}$

Proof. Remove the unwanted zero root:

- New variables $Y_0, \ldots, Y_n$
- New polynomial $\sum_i X_i Y_i - 1$ to the system. $\square$

Upper bounds for polynomial systems



Upper bounds

- PolSys($\mathbb{F}_p$) $\in$ PSPACE

  $\implies$ HomPolSys($\mathbb{F}_p$), Resultant($\mathbb{F}_p$) $\in$ PSPACE
- Under GRH, PolSys($\mathbb{Z}$) $\in$ AM [Koiran'96]

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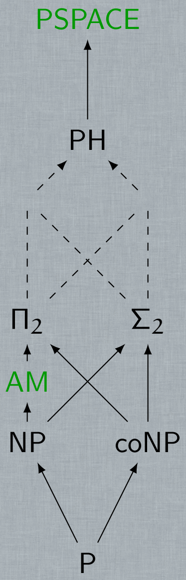

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Proof sketch of Koiran's result

- Let $f = (f_1, \dots, f_s)$, with $f_i \in \mathbb{Z}[X_1, \dots, X_n]$;
- Let $\mathcal{P}(x)$ be the set of prime numbers $\leqslant x$;
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There exist polynomial-time computable A and x_0 s.t.

- If f has no root in $\mathbb{C}$, then $\#\mathcal{P}_f(x_0) \leqslant A$;
- If f has a root in $\mathbb{C}$, then $\#\mathcal{P}_f(x_0) \geqslant 8A(\log A + 3)$.

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Algorithm.

1. Compute A, x_0;
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   - proba. 1 if f has a root in $\mathbb{C}$;
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	$\textsc{PolSys}$	$\textsc{HomPolSys}$	$\textsc{Resultant}$
$\mathbb{Z}$	NP-hard	NP-hard	NP-hard
$\mathbb{F}_p$	NP-hard	NP-hard	**Open**

Hardness in positive characteristics

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Idea of the reduction

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$$g_i := \sum_{j=1}^{s} \alpha_{ij} f_j, 0 \leqslant i \leqslant n.$$

- $\forall a \in \overline{\mathbb{F}_p}^{n+1} \left(\forall j, f_j(a) = 0 \boxed{\iff} \forall i, g_i(a) = 0 \right)$

if α_{ij} algebraically independent

- Replace algebraic independence by random choice

Two useful results

Effective Bertini Theorem

Let $f_1, \dots, f_s$ and $g_0, \dots, g_n$ be as on previous slide. Then there exists a polynomial F of degree at most 3^{n+1} s.t.

$$F(\boldsymbol{\alpha}) \neq 0 \implies \forall \boldsymbol{a} \big(\forall i, f_i(\boldsymbol{a}) = 0 \iff \forall j, g_j(\boldsymbol{a}) = 0\big).$$

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Lemma [DeMillo-Lipton, Zippel, Schwartz (1978-80)]

Let $F \in \mathbb{F}_q[X_0, \dots, X_n]$ be nonzero, of degree d. If $A_0, \dots, A_n$ are chosen independently at random in $\mathbb{F}_q$, then

$$\mathbb{P}[F(A_0, \dots, A_n) = 0] \leqslant \frac{d}{q}$$

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1. Build an extension $\mathbb{L}/\mathbb{F}_p$ with at least 3^{n+2} elements; [Shoup'90]

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Theorem [G.-Koiran-Portier'10-13]

Let p be a prime number. Resultant($\mathbb{F}_q$) is NP-hard for **degree-2** polynomials for some $q = p^s$, under **randomized reductions**.

Hardness in positive characteristics

- HomPolSys($\mathbb{F}_p$) is NP-hard:

  # homogeneous polynomials $\geqslant$ # variables

- Two strategies:
  - Reduce the number of polynomials
  - Increase the number of variables

HomPolSys

- Variables X_0 and $X_1, \ldots, X_n$ over $\mathbb{F}_p$
- Polynomials $X_0^2 - X_i^2$ for every $i > 0$ and
  - $X_0 \cdot (X_i + X_0)$
  - $X_0 \cdot (X_i + X_j)$
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- New variables: $Y_1, \dots, Y_{s-n-1}$

New system

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$$\alpha \text{ root of } f \implies (\alpha, 0) \text{ root of } g$$

Equivalence?

$(\mathbf{a}, \mathbf{b})$ non trivial root of $g \overset{?}{\Longrightarrow}$ $\mathbf{a}$ non trivial root of f

$$\begin{pmatrix} f_1(\mathbf{a}) & & \\ \vdots & & \\ f_n(\mathbf{a}) & & \\ f_{n+1}(\mathbf{a}) & & +\lambda b_1^2 \\ f_{n+2}(\mathbf{a}) & -b_1^2 & +\lambda b_2^2 \\ \vdots & & \\ f_{s-1}(\mathbf{a}) & -b_{s-n-2}^2 & +\lambda b_{s-n-1}^2 \\ f_s(\mathbf{a}) & -b_{s-n-1}^2 & \end{pmatrix}$$

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Last step

$$\det = \pm \left(\epsilon_1 + \epsilon_2\lambda + \cdots + \epsilon_N\lambda^{N-1}\right)$$

- Compute an irreducible polynomial $P \in \mathbb{F}_p[\xi]$ of degree N;
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Let p be a prime number.

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Theorem [G.-Koiran-Portier'10-13]

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