Determinantal representations of polynomials

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Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

- Complexity of the determinant

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

- Complexity of the determinant
- Determinant vs. Permanent: Algebraic “$P = NP$?”

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

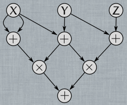
- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "$P = NP?$"
- Links between circuits, ABPs and the determinant

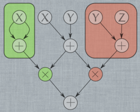
Determinantal representations

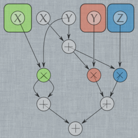
$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

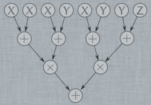
- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "$P = NP$?"
- Links between circuits, ABPs and the determinant
- Convex optimization

Circuits

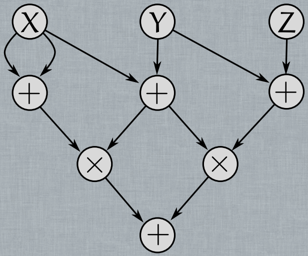








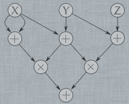
$$2X(X+Y)+(X+Y)(Y+Z)$$

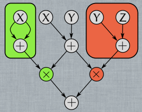


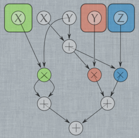
Arithmetic circuit

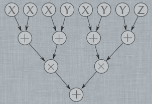
Size	6
Inputs	3

Circuits

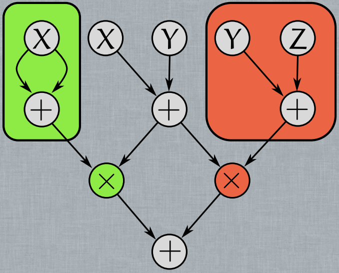








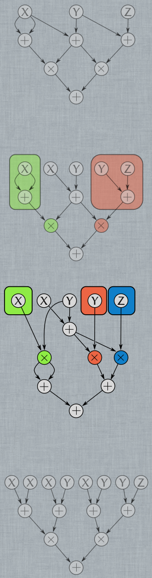
$$2X(X+Y)+(X+Y)(Y+Z)$$



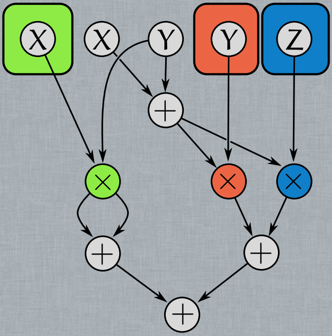
Weakly-skew circuit

Size	6
Inputs	5

Circuits



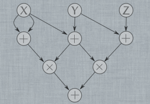
$$2X(X+Y)+(X+Y)(Y+Z)$$

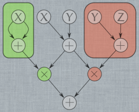


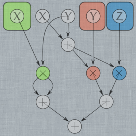
Skew circuit

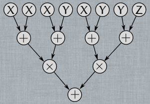
Size	7
Inputs	5

Circuits

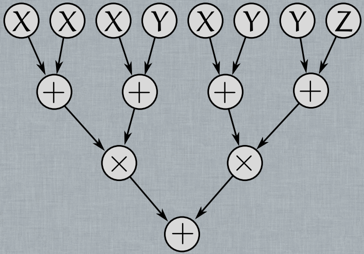








$$2X(X+Y)+(X+Y)(Y+Z)$$



Formula

Size	7
Inputs	8

Results

Proposition

- Formula of **size** $s \rightsquigarrow$ Determinant of a matrix of **dimension** $(s+2)$ [Valiant'79]

Results

Proposition

- Formula of **size** s $\rightsquigarrow$ Determinant of a matrix of **dimension** $(s+1)$ [Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11]

Results

Proposition

- Formula of **size** $s \rightsquigarrow$ Determinant of a matrix of **dimension** $(s+1)$ [Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11]
- Weakly-skew circuit of **size** s with i **inputs** $\rightsquigarrow$ Determinant of a matrix of **dimension** $(s+i+1)$ [Toda'92, Malod-Portier'08]

Results

Proposition

- Formula of **size** s $\rightsquigarrow$ Determinant of a matrix of **dimension** $(s+1)$ [Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11]
- Weakly-skew circuit of **size** s with i **inputs** $\rightsquigarrow$ Determinant of a matrix of **dimension** $(s+i+1)$ [Toda'92, Malod-Portier'08]
- Determinant $(n \times n)$ $\rightsquigarrow$ Skew circuit of **size** $\frac{1}{2}n^4 + o(n^4)$ [Mahajan-Vinay'97]

Results

Proposition

- Formula of **size** $s \rightsquigarrow$ Determinant of a matrix of **dimension** $(s+1)$ [Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11]
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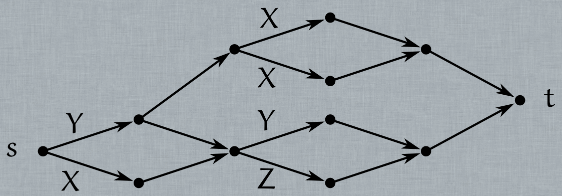

Algebraic Branching Program

From Formulas to Branching Programs

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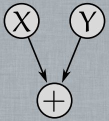

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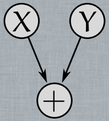

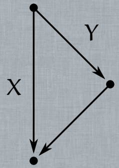

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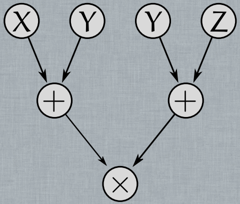

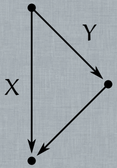

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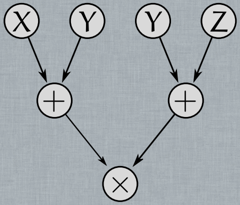

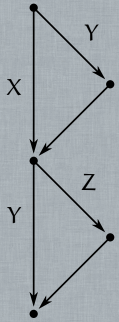

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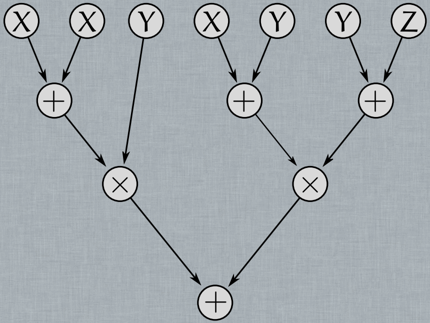

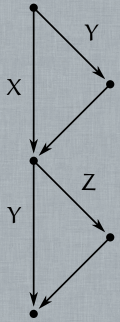

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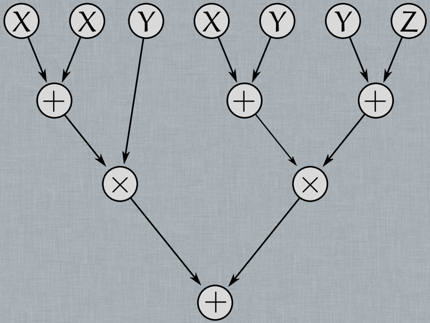

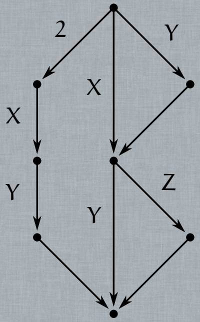

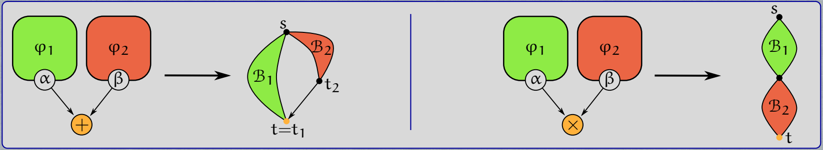

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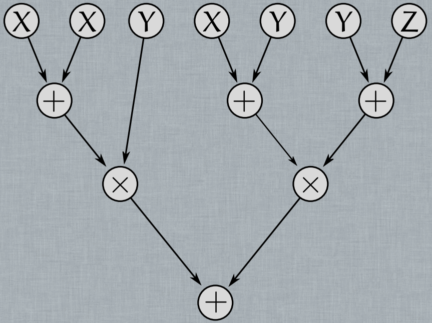
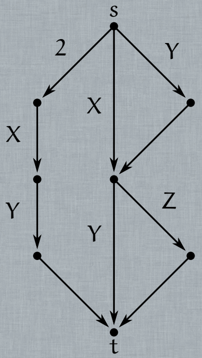
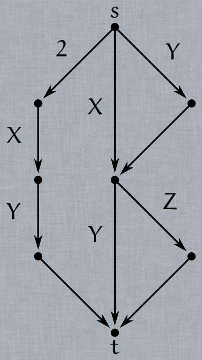
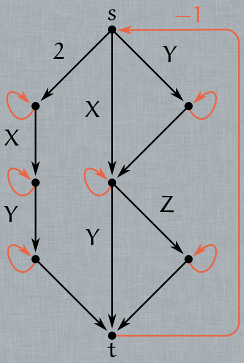




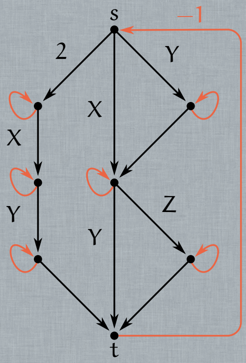
From Branching Programs to Determinants



From Branching Programs to Determinants

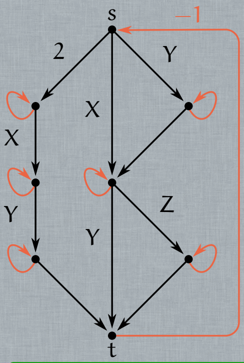


From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

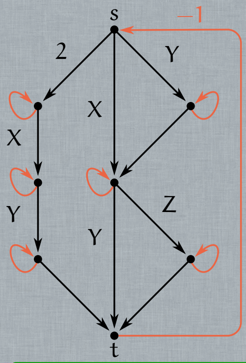
From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det M = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} M_{i,\sigma(i)}$$

From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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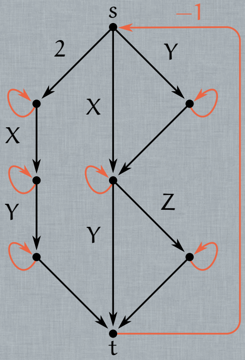
- **Cycle covers** $\iff$ **Permutations**

From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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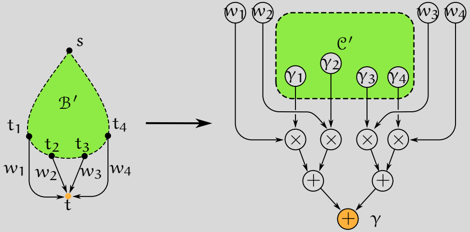
From Branching Programs to Skew Circuits



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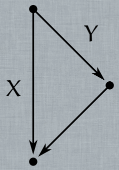

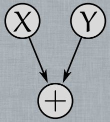

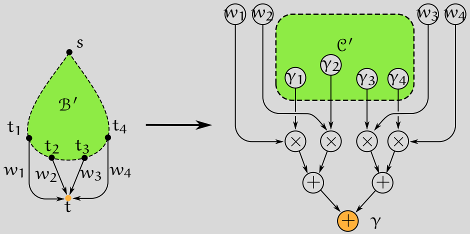

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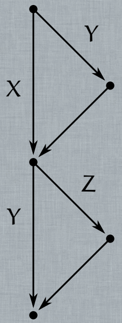

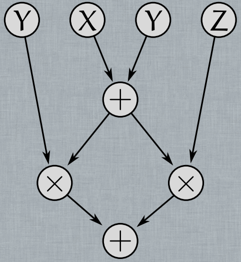

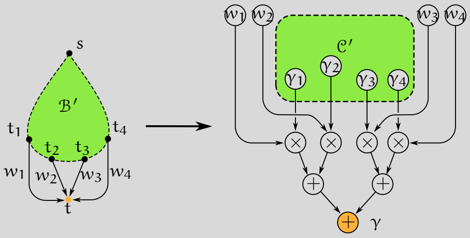

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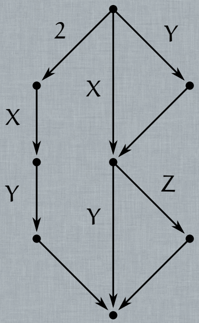

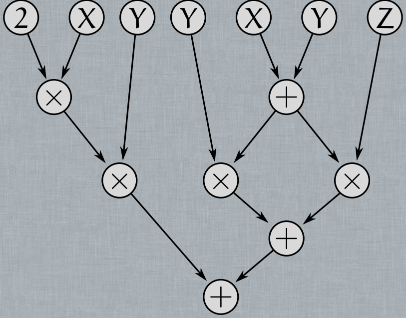

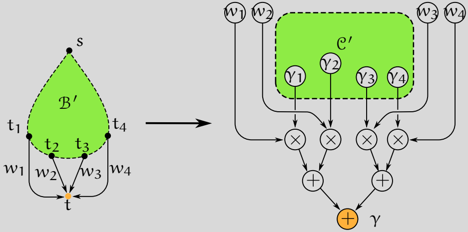

Clows and the determinant

Definition

Let $G = (V, A)$, $V = \{1, \dots, n\}$.

- Clow C: $v_1 \to v_2 \to \cdots \to v_\ell \to v_{\ell+1} = v_1$
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Theorem [Mahajan-Vinay'97]

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Proof idea. If $\mathcal{C}$ is not made of disjoint cycles, cut a clow into two smaller clows or merge two clows

Branching Program for the Determinant

(u, v, i)

- u: head of current clow
- v: current vertex
- i: number of visited arcs

$(u, u, i)^\star$
End of clow of head u

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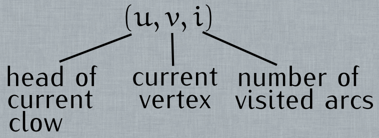

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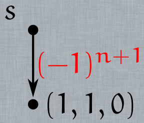

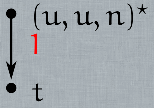

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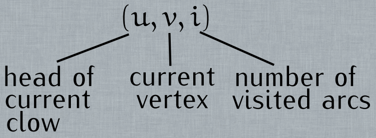

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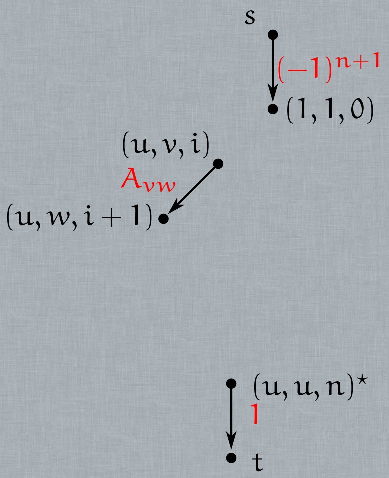

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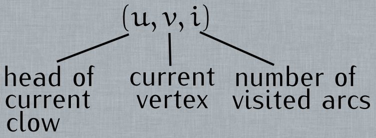

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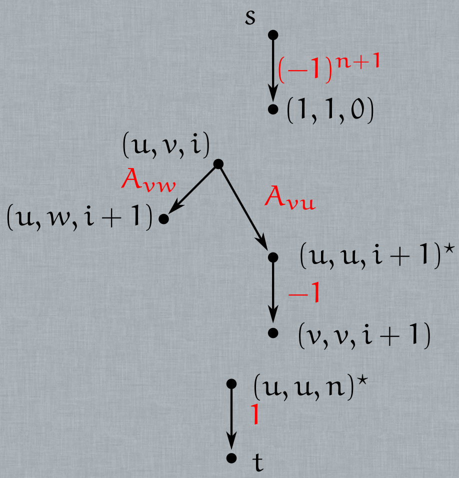

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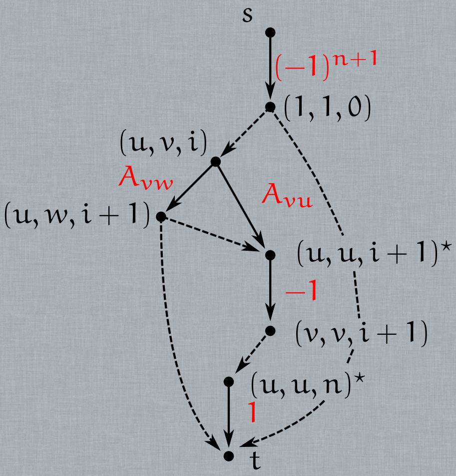

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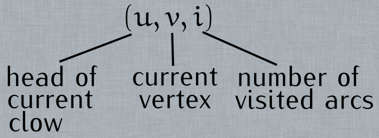

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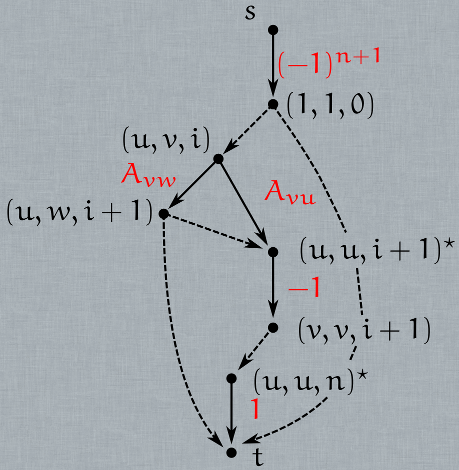

Theorem [Mahajan-Vinay'97,G.'12]

There exists a branching program of size $\frac{1}{3}n^3 + o(n^3)$ for the determinant $(n \times n)$, with $\frac{1}{4}n^4 + o(n^4)$ arcs.

Branching Program for the Permanent

$$\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

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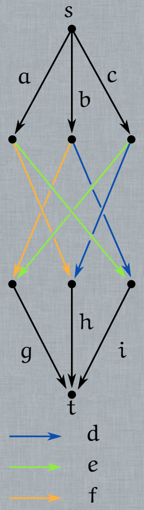
Theorem [G.'12]

There exists a **branching program of size 2^n** representing the **permanent of dimension n**.



Permanent versus Determinant

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Conjecture [Algebraic $\mathsf{P} \neq \mathsf{NP}$]

The **permanent of dimension n** is **not** a projection of the **determinant of dimension $N = 2^{o(n)}$**.

Note. Best known lower bound: $n^2/2$ [Mignon-Ressayre'04]

Results

Proposition

- Formula of size $s \rightsquigarrow$ Determinant of a matrix of dimension $(s+1)$ [Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11]
- Weakly-skew circuit of size s with i inputs $\rightsquigarrow$ Determinant of a matrix of dimension $(s+i+1)$ [Toda'92, Malod-Portier'08]
- Determinant $(n \times n) \rightsquigarrow$ Skew circuit of size $\frac{1}{2}n^4 + o(n^4)$ [Mahajan-Vinay'97]

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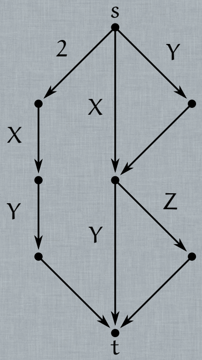
From Branching Programs to Symmetric Determinants



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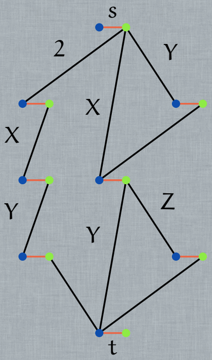

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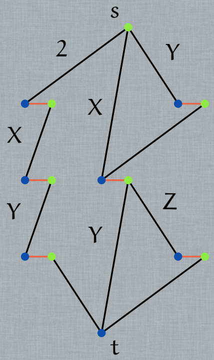

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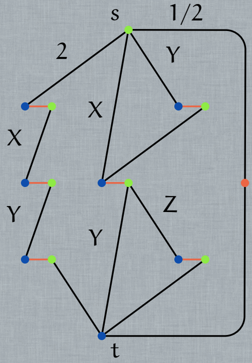

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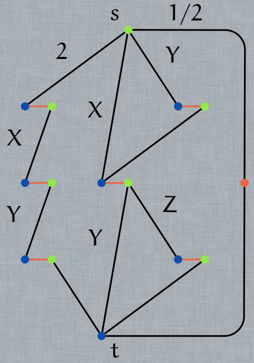

$$S = \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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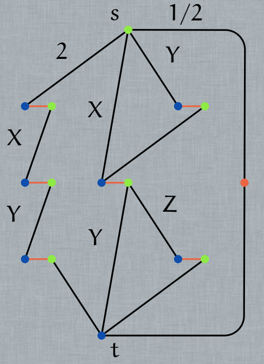

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Theorem [G., Monteil, Thomassé'13]

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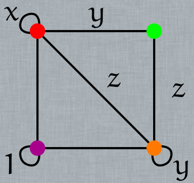

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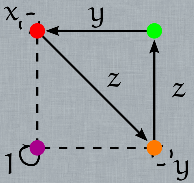

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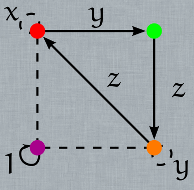

SDR in characteristic 2

Theorem [G., Monteil, Thomassé'13]

There are polynomials **without SDR** in characteristic 2, e.g. $xy+z$.

- A polynomial is said **representable** if it has an SDR.

Determinant in characteristic 2 of symmetric matrices

$\mathfrak{I}_n$ = Involutions of $\{1,\dots,n\}$

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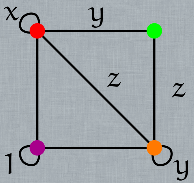

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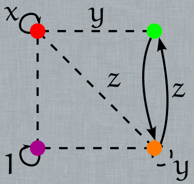

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Lemma

- P and Q are representable $\implies$ $P \times Q$ is representable.

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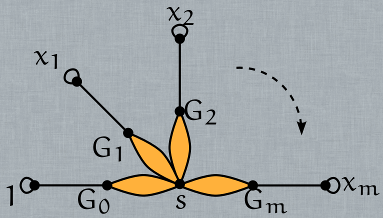
$L(x_1, \ldots, x_m) = P_0^2 + x_1 P_1^2 + \cdots + x_m P_m^2$ is representable.



Obstructions to representability

Theorem

If P is representable, then

$$P \equiv L_1 \times \cdots \times L_k \bmod \langle x_1^2 + 1, \ldots, x_m^2 + 1 \rangle$$

where the L_i's are linear. (linear = degree-1)

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If P is representable, then

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Such a P is said **factorizable** *modulo* $\langle x_1^2 + \ell_1^2, \dots, x_m^2 + \ell_m^2 \rangle$.

Multilinear polynomials

Theorem

Let P be a multilinear polynomial. The following propositions are equivalent:

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Finding a factor

Is $xy + z$ representable?

$$(x+y+z+1) \times (x+y+z+1) \times \cdots \times (x+y+z+1)$$

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Theorem

Under *suitable* conditions, P is factorizable if and only if

$$P \equiv \mathrm{lin}(P) \times \frac{1}{\alpha_i} \frac{\partial P}{\partial x_i} \mod \langle x_1^2, \ldots, x_m^2 \rangle,$$

where $\alpha_i x_i$ is a monomial of $\mathrm{lin}(P)$.

Conclusion

Same **expressiveness**:

- (Weakly-)Skew circuits

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Main open question [Algebraic "P = NP?"]

What is the **smallest** N s.t. the **permanent of dimension** n is a projection of the **determinant of dimension** N?