Root finding over finite fields using Graeffe transforms

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- Building block for many algorithms in computer algebra: root finding over $\mathbb{Z}$, factorization, sparse interpolation, ...
- Applications in cryptography, error correcting codes, ...
- Derandomization
- Sparse interpolation: bottleneck in practice

[van der Hoeven & Lecerf, 2014]

Theorem

The roots of $f \in \mathbb{F}_q[X]$ can be computed in **deterministic time** $\mathrm{poly}(q, d)$, where $d = \deg(f)$.

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- Better complexity bounds when $q-1$ is *smooth*

  [Moenck (1977), von zur Gathen (1987), Mignotte-Schnorr (1988), Rónyai (1989), Shoup (1992), Źrałek (2010), ...]

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$$f(X) = \prod_{i=1}^{d} (X - \alpha_i), \quad \alpha_i \in \mathbb{F}_q^*, \quad \alpha_i \neq \alpha_j \qquad \text{(A)}$$

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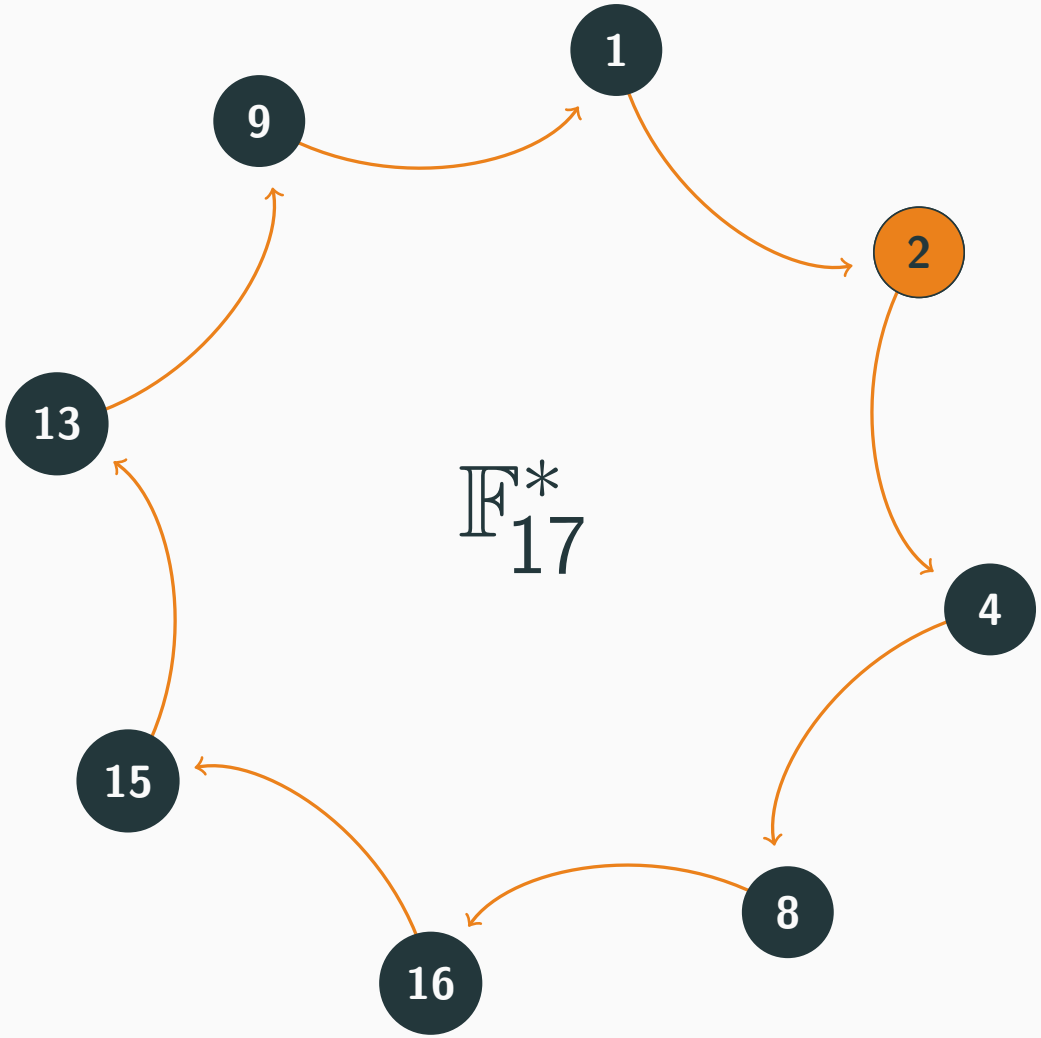
- **FFT finite field**: $\mathbb{F}_p$ with $p = M \cdot 2^m + 1$ and $M = O(\log p)$
  - Old algorithms revisited
  - New technique based on **Graeffe transforms**
  - Fast implementations

Multiplicative structure of finite fields



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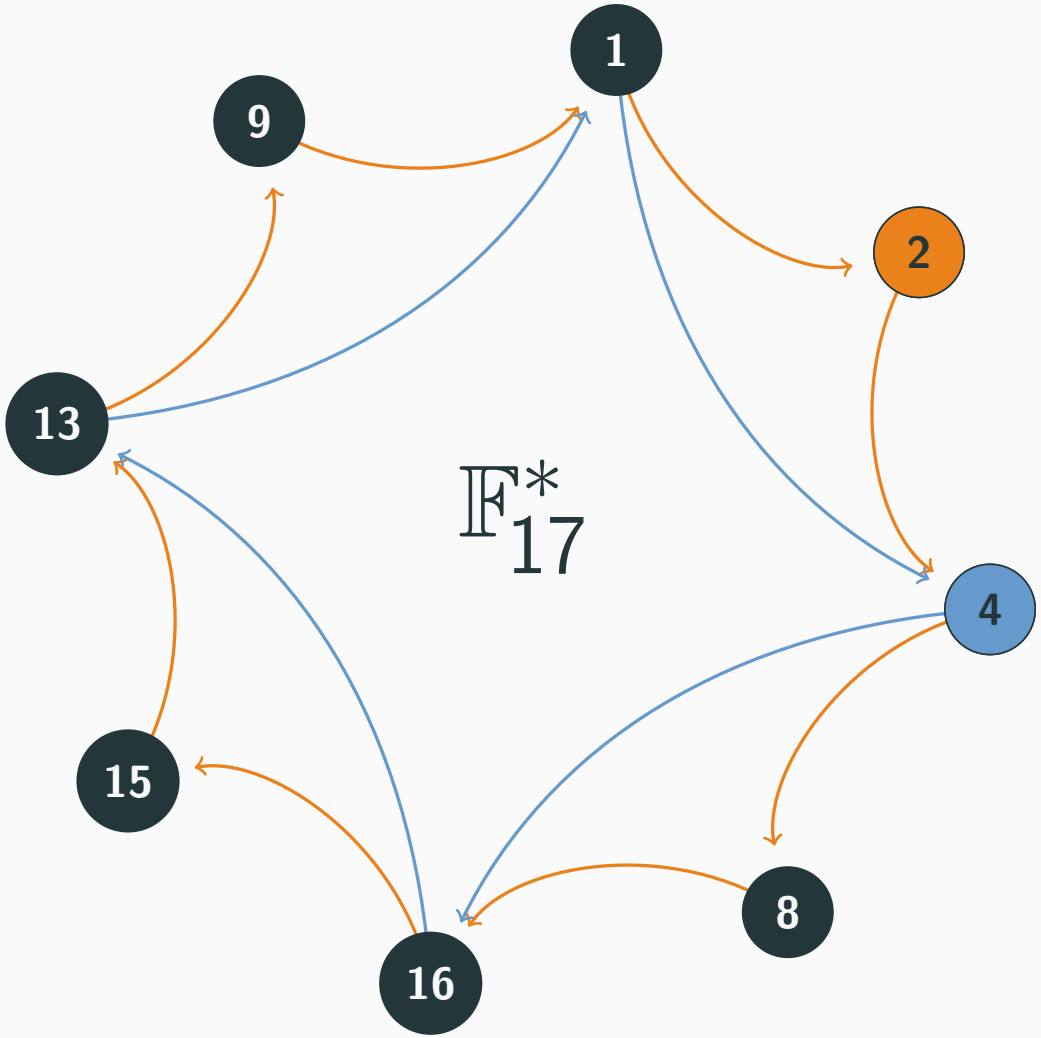

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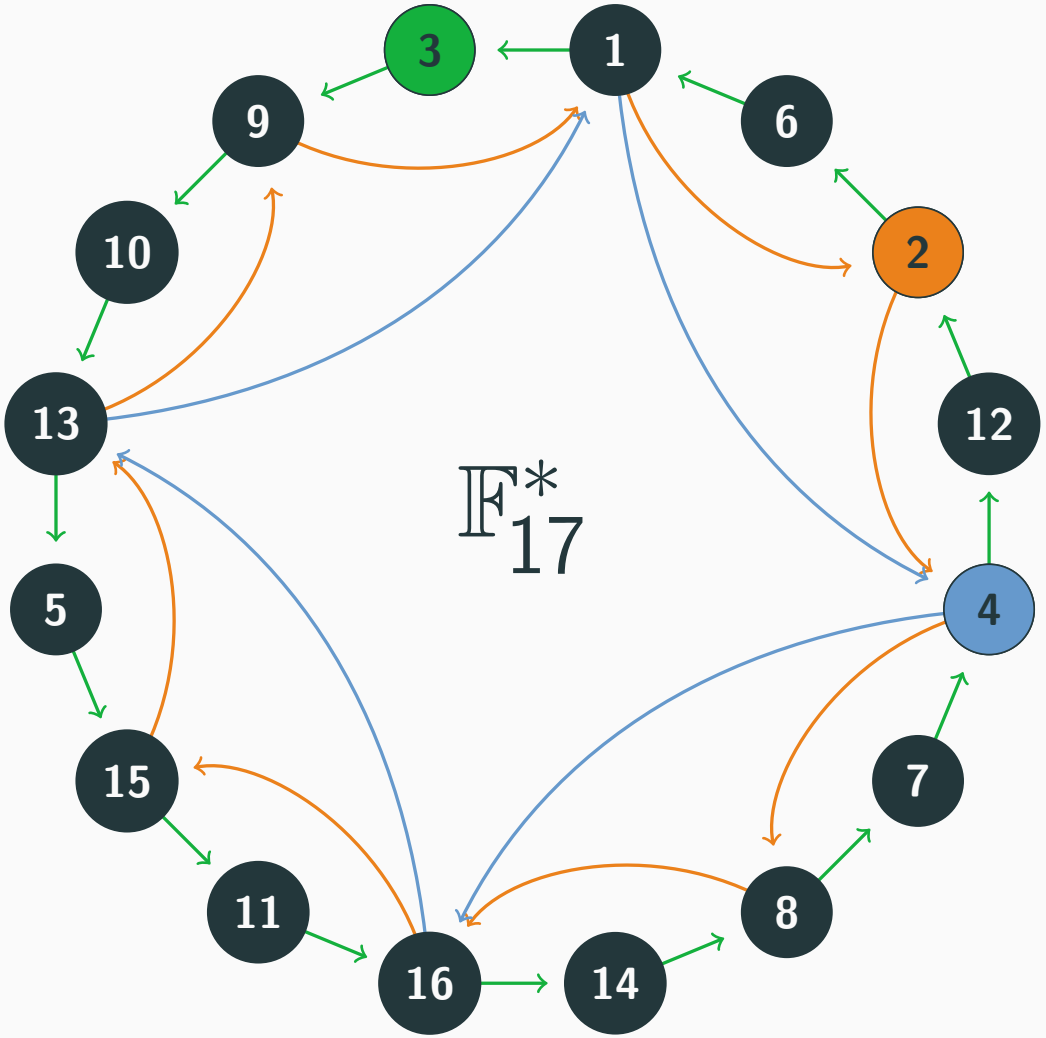

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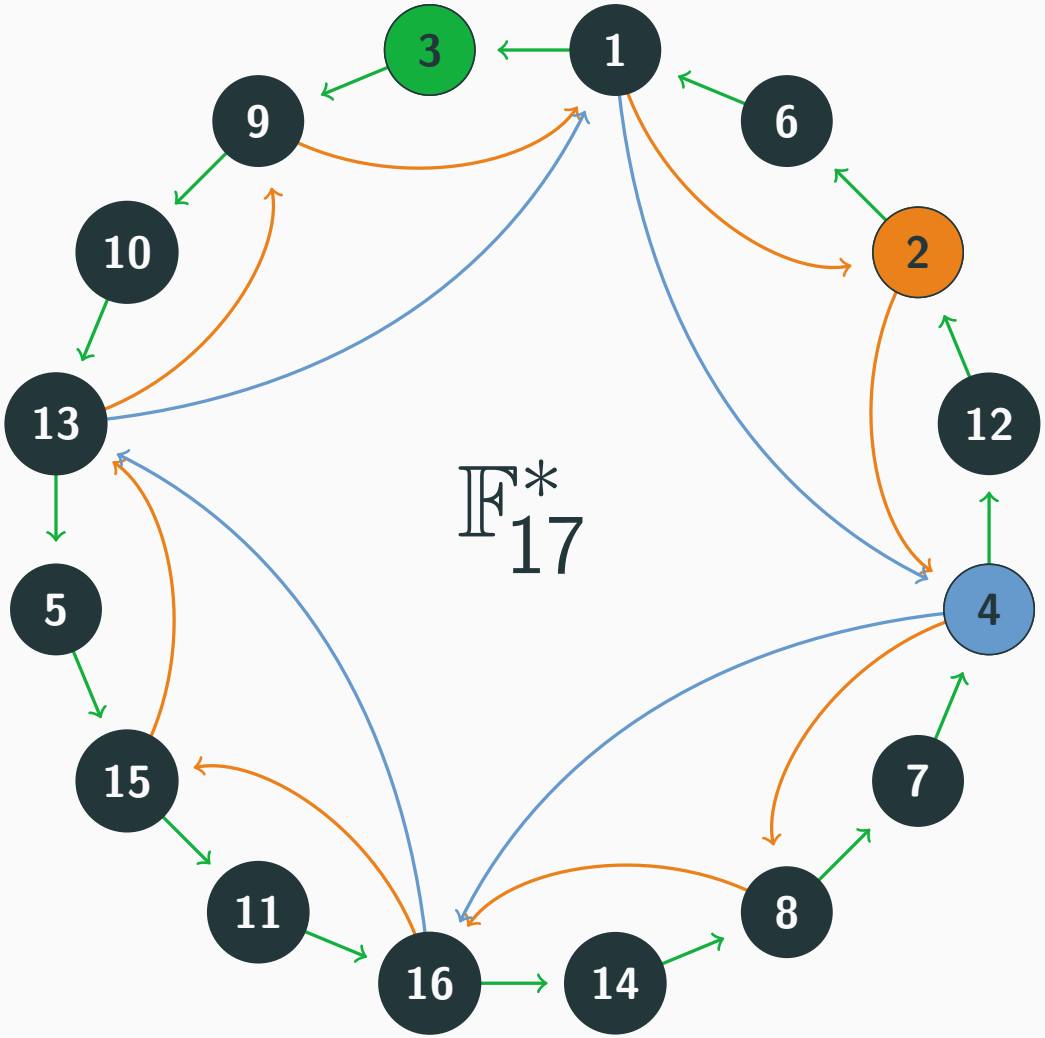

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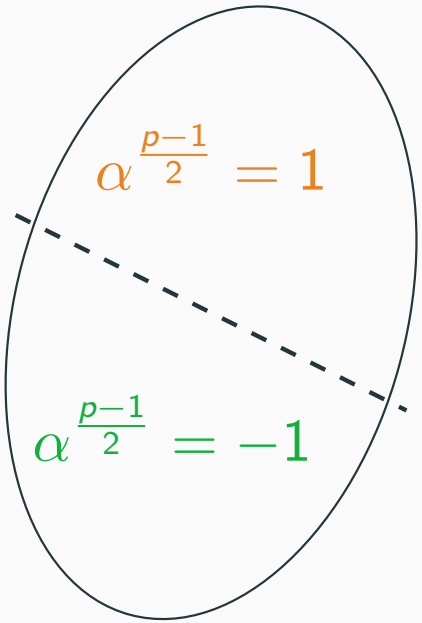

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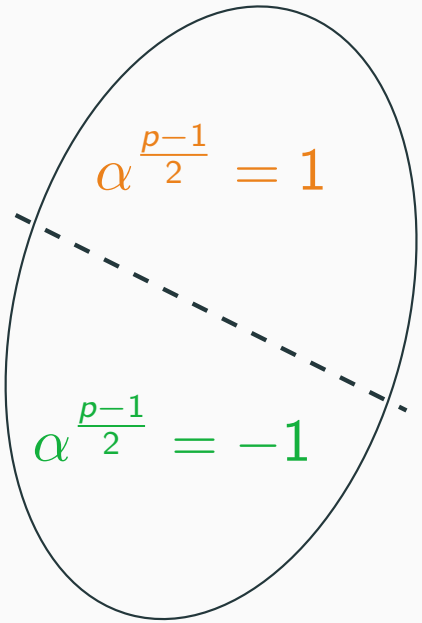

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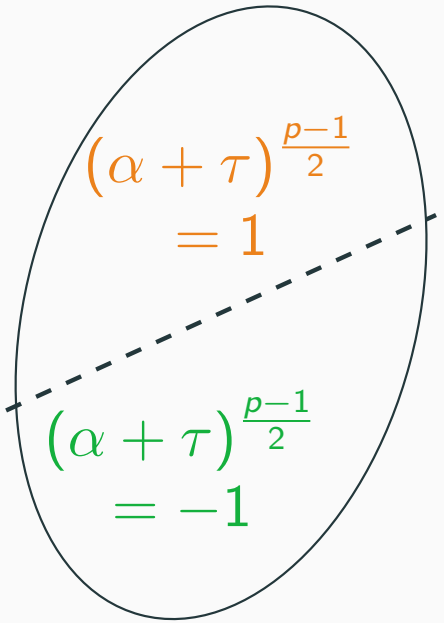

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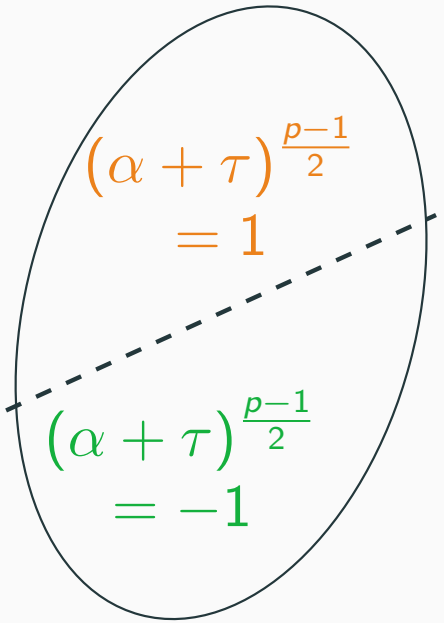

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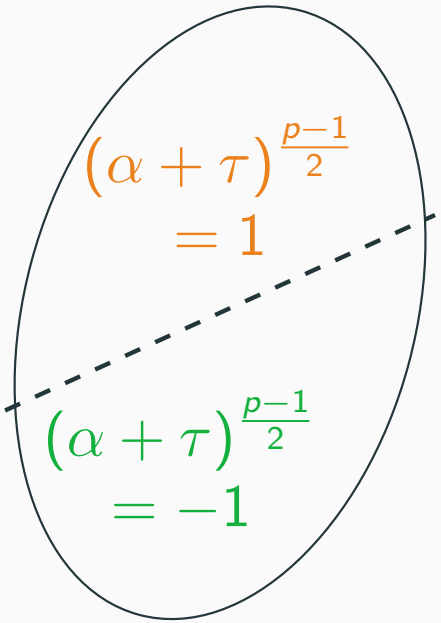

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Randomized algorithm

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Modified Rabin's algorithm (for FFT finite fields)

$$X^{p-1} - 1 = \prod_{i=0}^{2^\ell - 1} (X^{M2^{m-\ell}} - \xi^i), \text{ where } \xi \text{ is primitive of order } 2^\ell.$$

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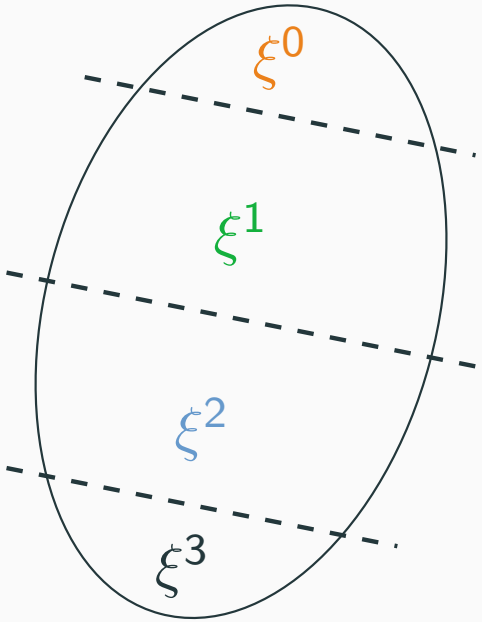

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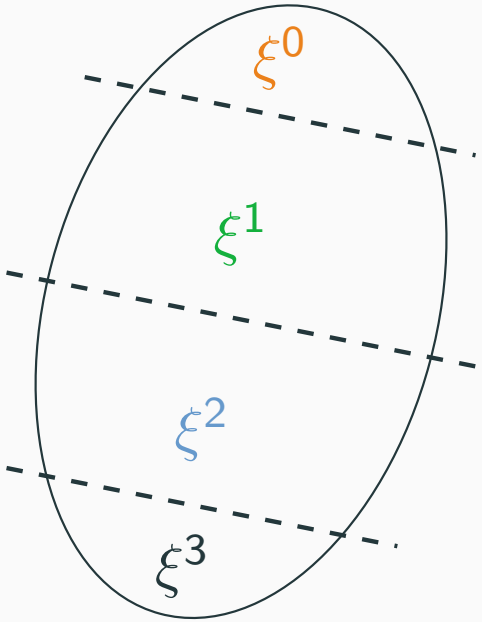

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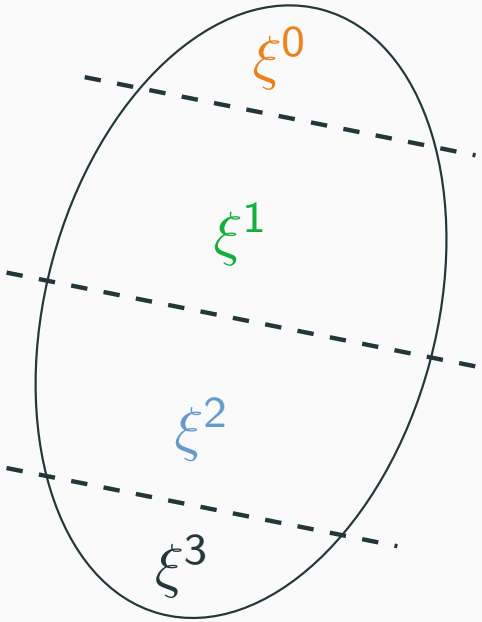

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Worthwhile in practice for **small** $\ell = 2, 3, \ldots$

New technique: Graeffe transform

Let $f(X) = \prod_i (X - \alpha_i)$

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Remarks:

- $G_{\rho_1\rho_2} = G_{\rho_1} \circ G_{\rho_2}$
- $G_{q-1}(f)(X) = \prod_i (X - \alpha_i^{q-1}) = (X-1)^d$

Lemma

Let ξ be a primitive root of unity of order π. Then

$$G_\pi(f)(X^\pi) = f(X)f(\xi X)\cdots f(\xi^{\pi-1}X).$$

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Theorem

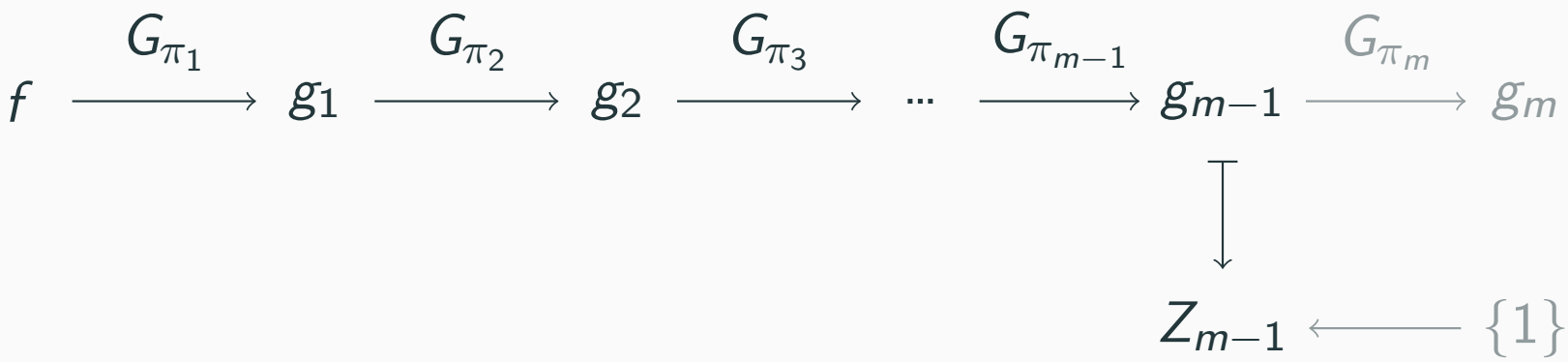
Let $f \in \mathbb{F}_q[X]$ and $q-1 = \pi_1 \cdots \pi_m$. The Graeffe transforms $G_{\pi_1}(f)$, $G_{\pi_1\pi_2}(f)$, ..., $G_{\pi_1\cdots\pi_{m-1}}(f)$ can be computed in time $(d\log^2 q)^{1+\delta}$ for all $\delta > 0$.

Based on modular composition [Kedlaya-Umans (2011)].

Using Graeffe transforms

$$f \xrightarrow{G_{\pi_1}} g_1 \xrightarrow{G_{\pi_2}} g_2 \xrightarrow{G_{\pi_3}} \cdots \xrightarrow{G_{\pi_{m-1}}} g_{m-1} \xrightarrow{G_{\pi_m}} g_m$$

Using Graeffe transforms



- $Z_{m-1} \subseteq \{\xi^i : 0 \leq i < \pi_m\}$ (ξ: primitive root of order π_m)
  $= \{\zeta^{(q-1)i/\pi_m} : 0 \leq i < \pi_m\}$ (ζ: primitive element of $\mathbb{F}_q^*$)

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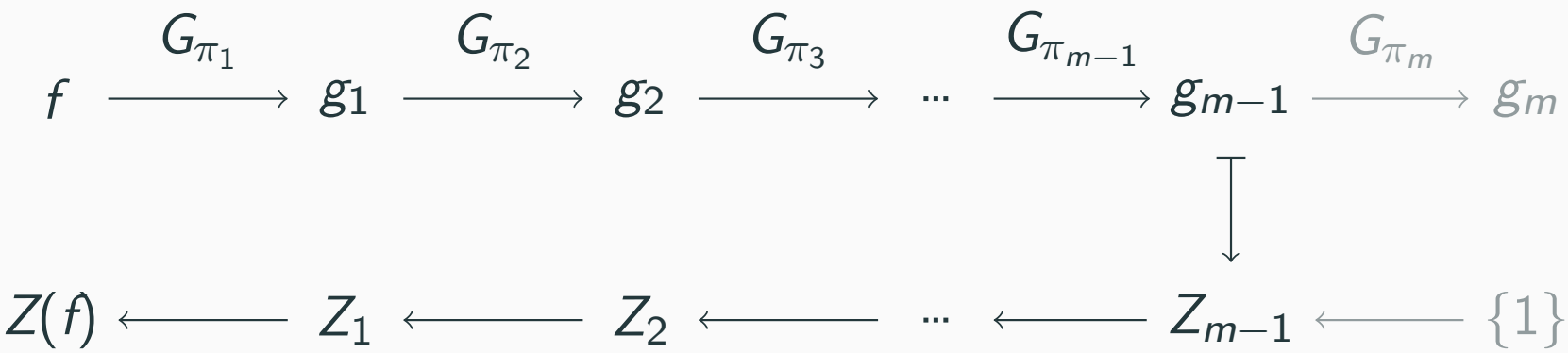

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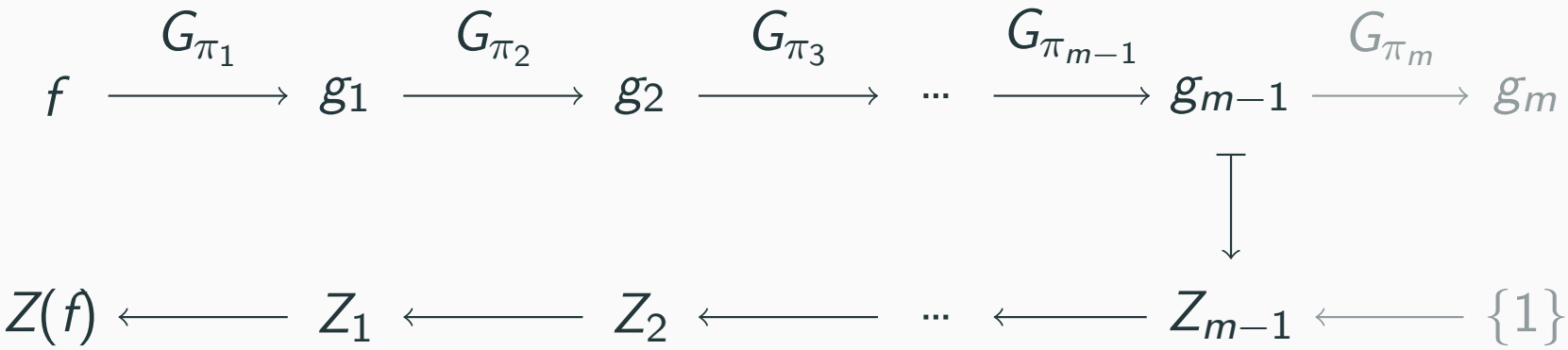

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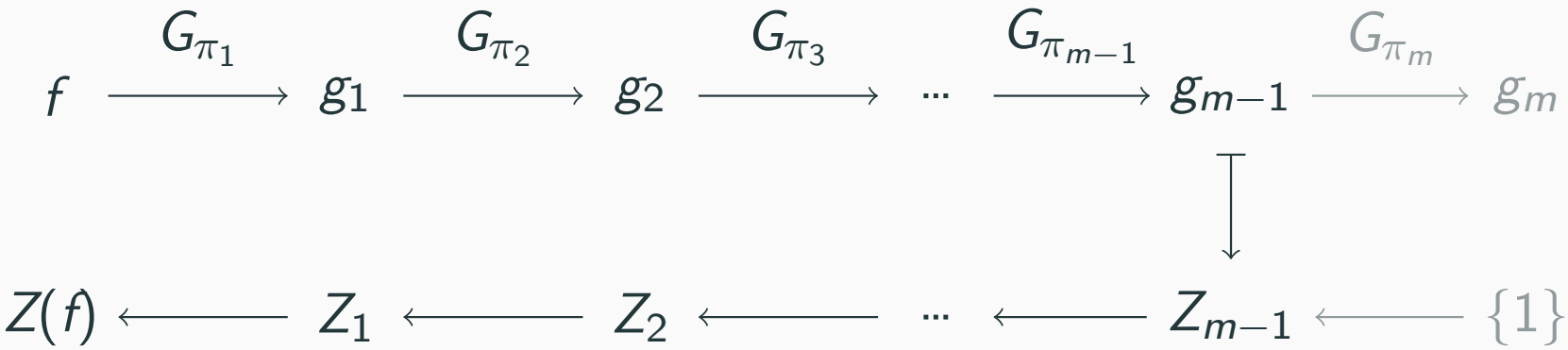

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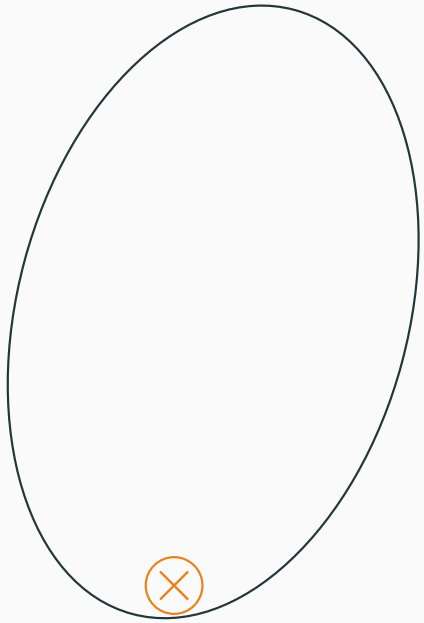
- If $\beta = \zeta^e$, $\alpha \in \left\{\zeta^{\frac{e+(q-1)i}{\pi_k}} : 0 \leq i < \pi_k\right\}$

A deterministic algorithm



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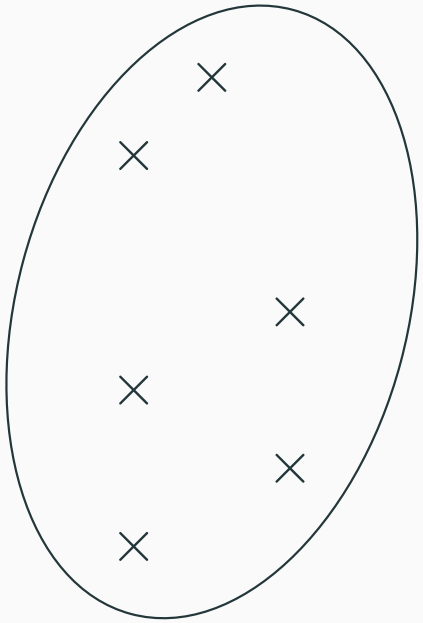

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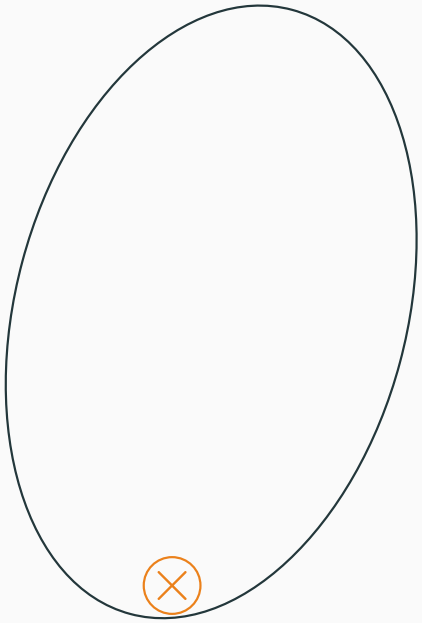

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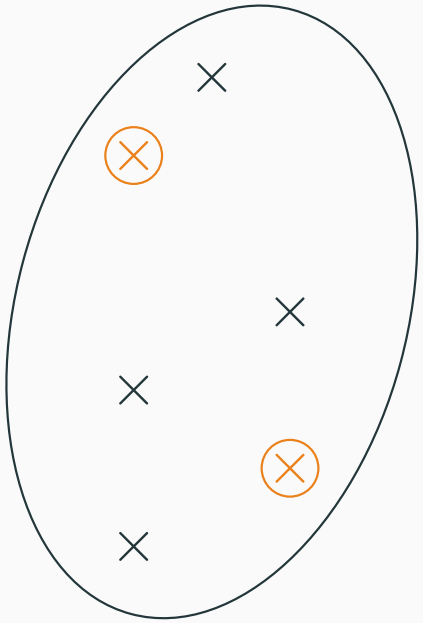

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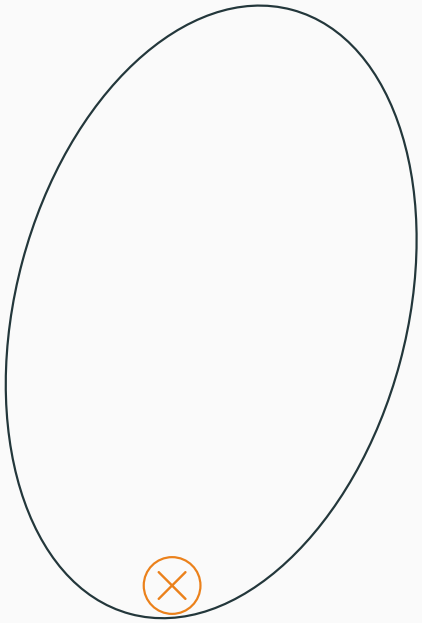

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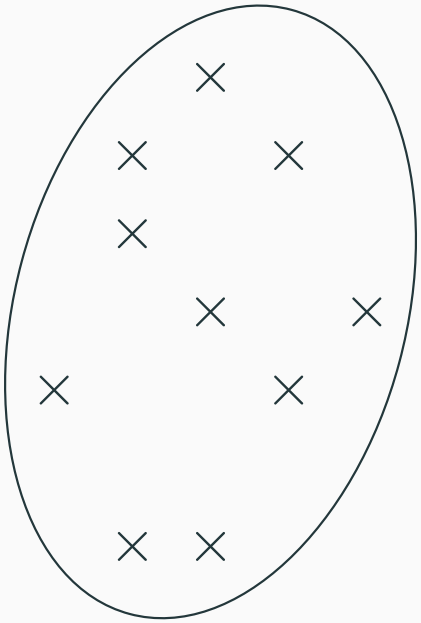

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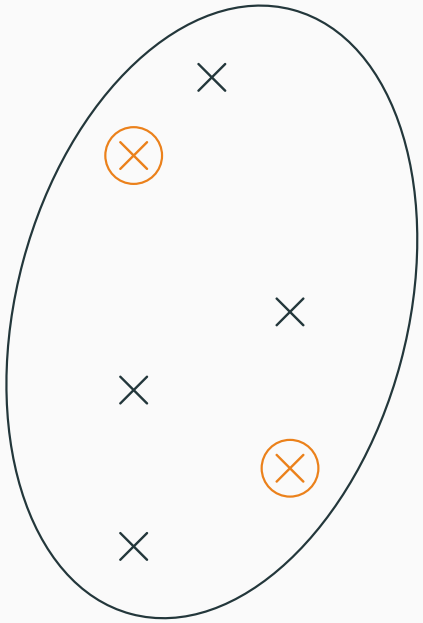

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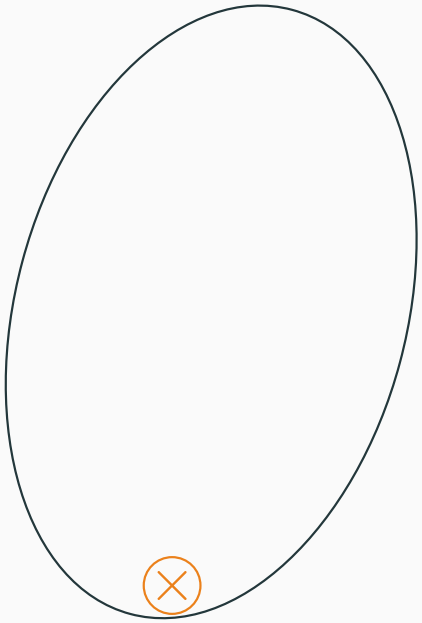

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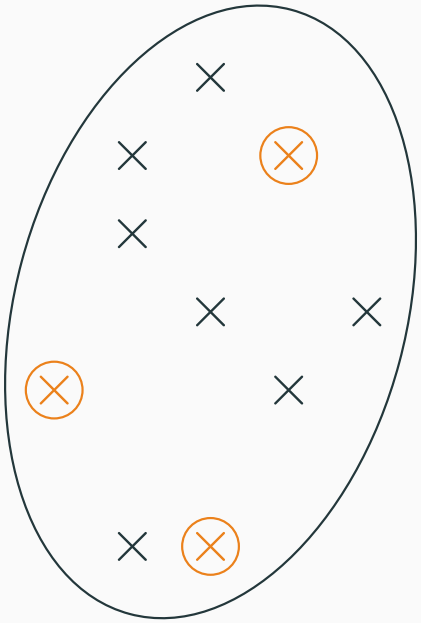

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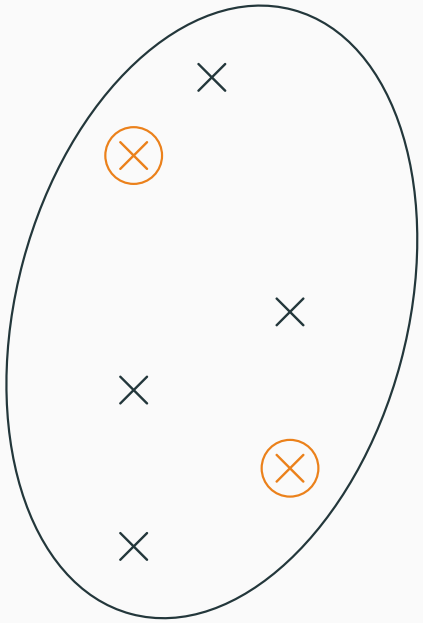

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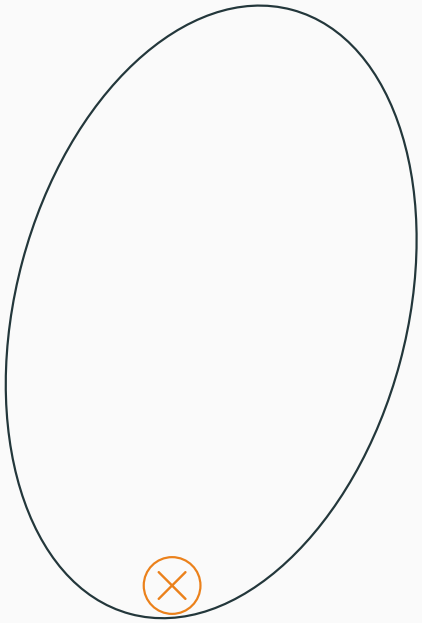

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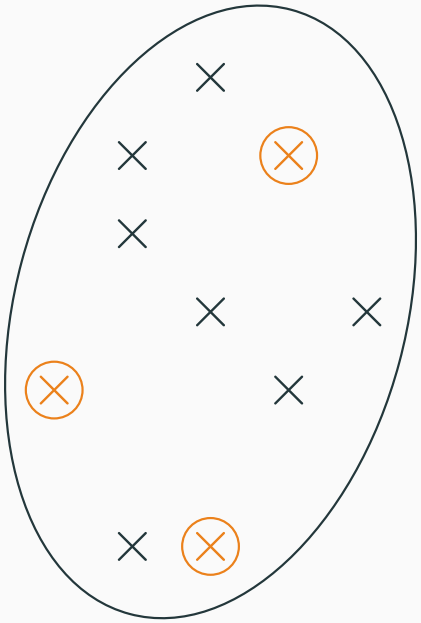

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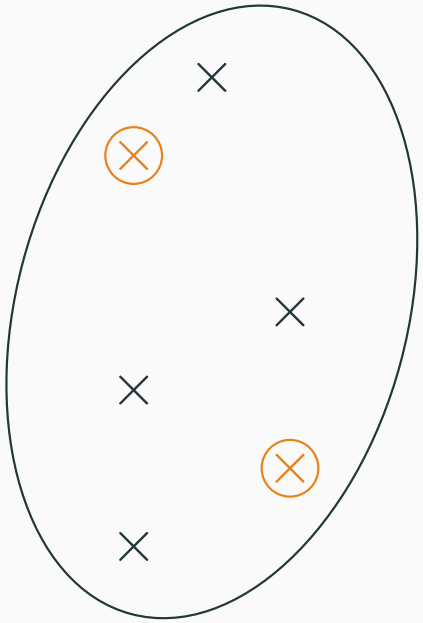

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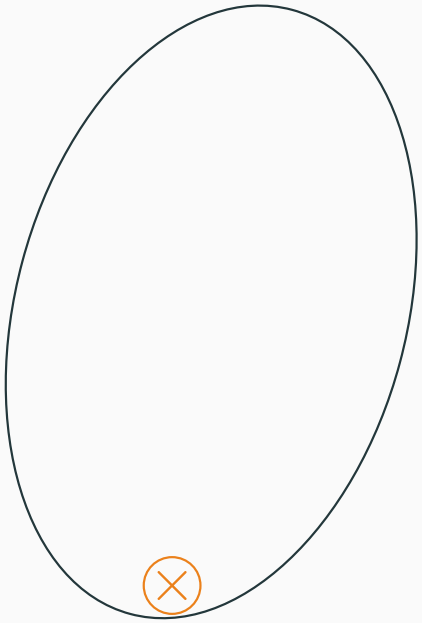

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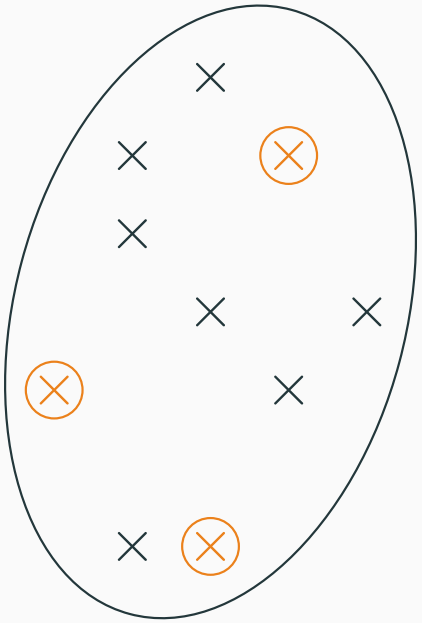

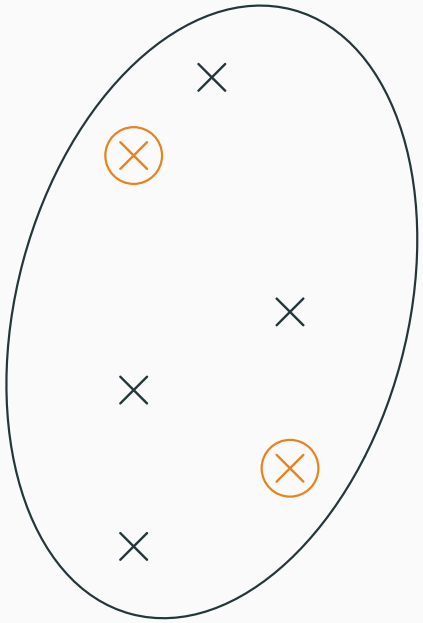

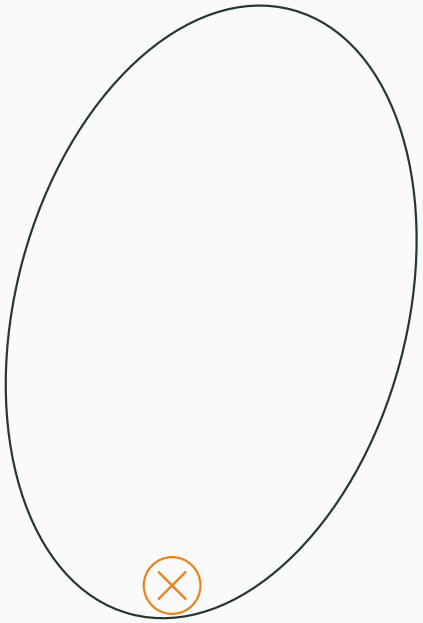

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Theorem

If $\max_i \pi_i = O(\log q)$, the algorithm runs in time $\tilde{O}(d \log^3 q)$.

Deterministic complexity

Lemma

Given $g = G_\pi(f)$ and the roots $\{\beta_1, \ldots, \beta_k\}$ of g, one can compute the roots of f in time $\tilde{O}(\sqrt{\pi} d \log q) + (d \log q)^{1+\delta}$ for all $\delta > 0$.

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Ingredients:

1. Factor $f = \prod_i f_i$ s.t. for all i, $G_\pi(f_i) = (X - \beta_i)^{m_i}$

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Theorem

Given $f \in \mathbb{F}_q[X]$ satisfying (A), the factorization of $q-1$ and a primitive element of $\mathbb{F}_q^*$, the roots of f can be computed in time

$$\tilde{O}(\sqrt{S(q-1)}d\log^2 q) + (d\log^2 q)^{1+\delta}$$

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- Refine Shoup's complexity bounds

Randomization

$\mathbb{F}_p$ **with** $p = M \cdot 2^m + 1$

Definition

The **tangent Graeffe transform of order** π of $f \in \mathbb{F}_p[X]$ is

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Remarks:

- $(f + \varepsilon f')(X) = f(X + \varepsilon)$
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Lemma

Let $g + \varepsilon \overline{g} = G_{2^\ell}(f + \varepsilon f')$. **A root β of g is simple iff** $\overline{g}(\beta) \neq 0$.
The corresponding root of f is $\alpha = 2^\ell \beta g'(\beta)/\overline{g}(\beta)$.

Goal: Ensure many simple roots

- Replace f by $f_\tau(X) = f(X + \tau)$ for a random $\tau \in \mathbb{F}_p$.

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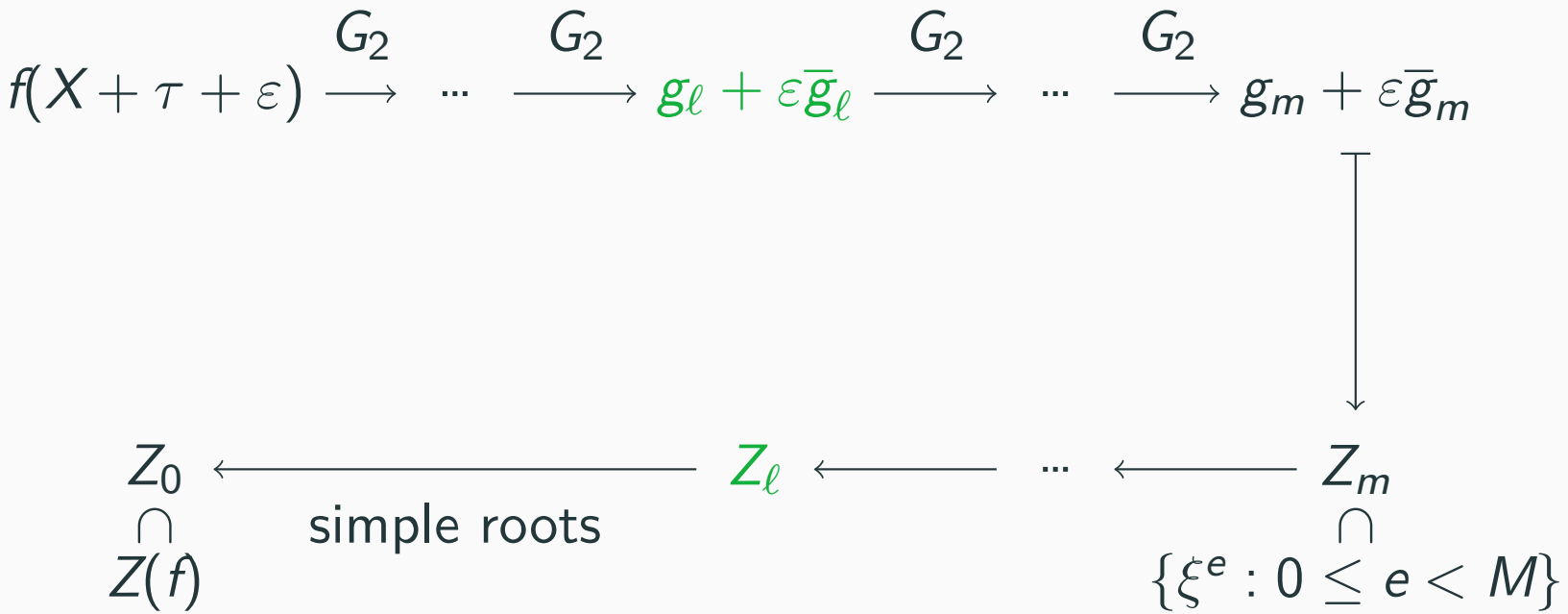

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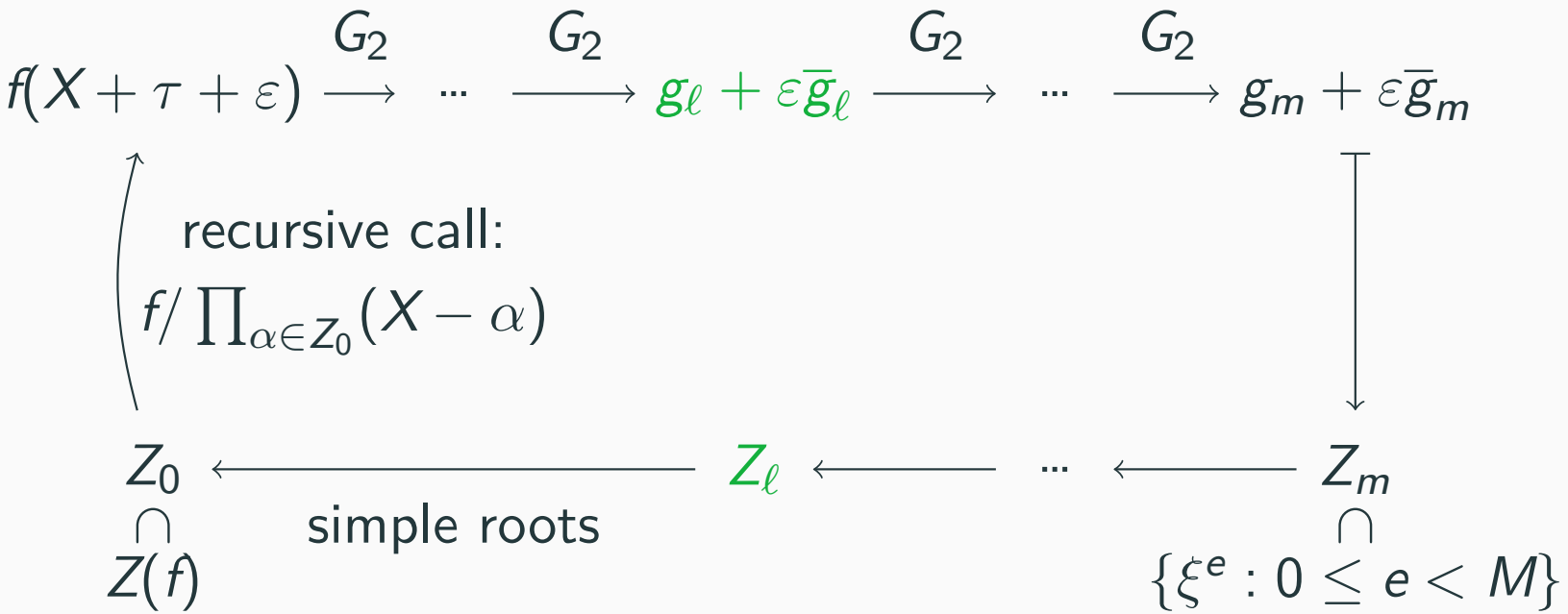

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Given $f \in \mathbb{F}_p[X]$ satisfying (A) and a primitive element of $\mathbb{F}_p^*$, the randomized algorithm runs in **expected time** $\tilde{O}(d \log^2 p)$, for $p = M \cdot 2^m + 1$ with $M = O(\log p)$.

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- Same asymptotic as Rabin's algorithm
- Better efficiency in practice (gcd $\rightsquigarrow$ multipoint evaluation)
- Primitive elements easy to compute in practice

Heuristic version

$\mathbb{F}_p$ **with** $p = M \cdot 2^m + 1$

Heuristic

If $2^\ell \simeq p/d$, $G_{2^\ell}(f(X+\tau))$ has $\Omega(d)$ **simple roots** with probability $\geq 1/2$, for a random $\tau \in \mathbb{F}_p$.

Justification: holds for a random f rather than $f(X+\tau)$.

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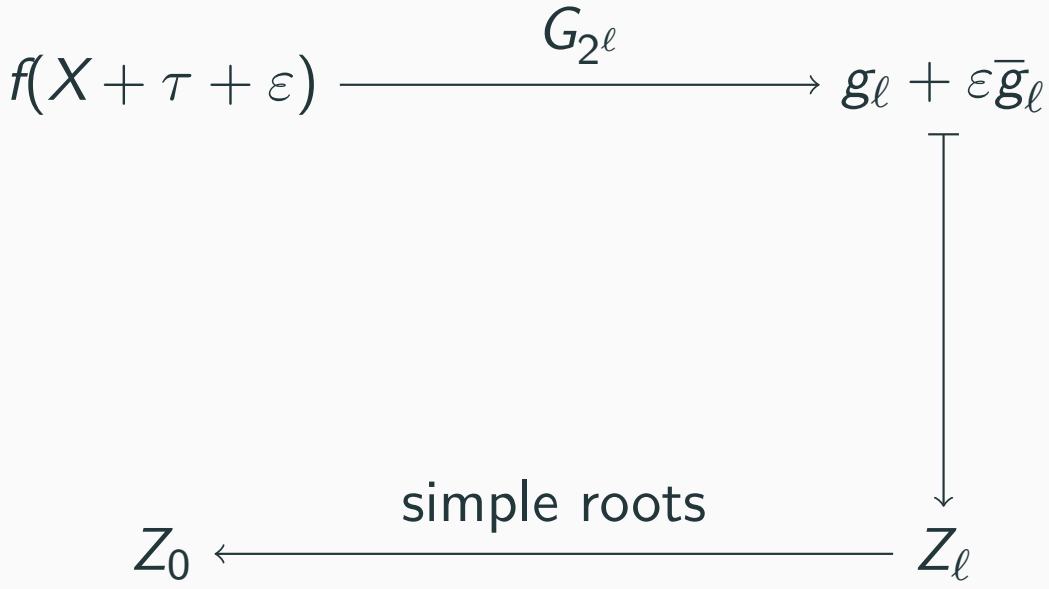

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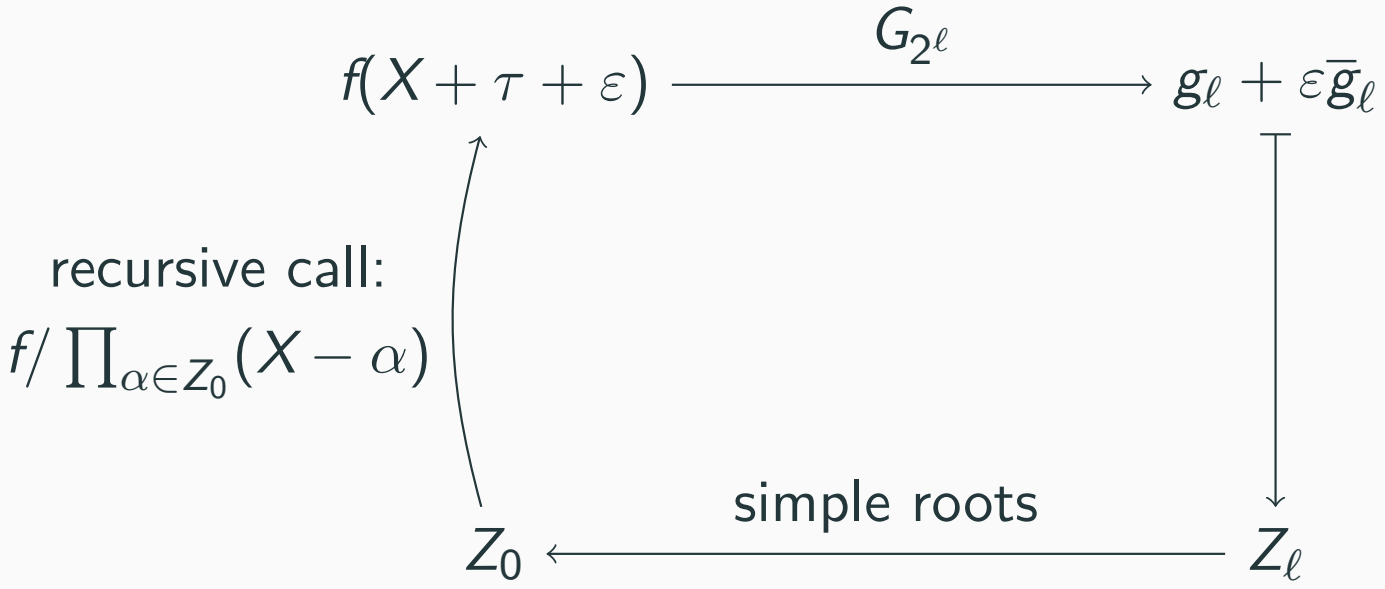
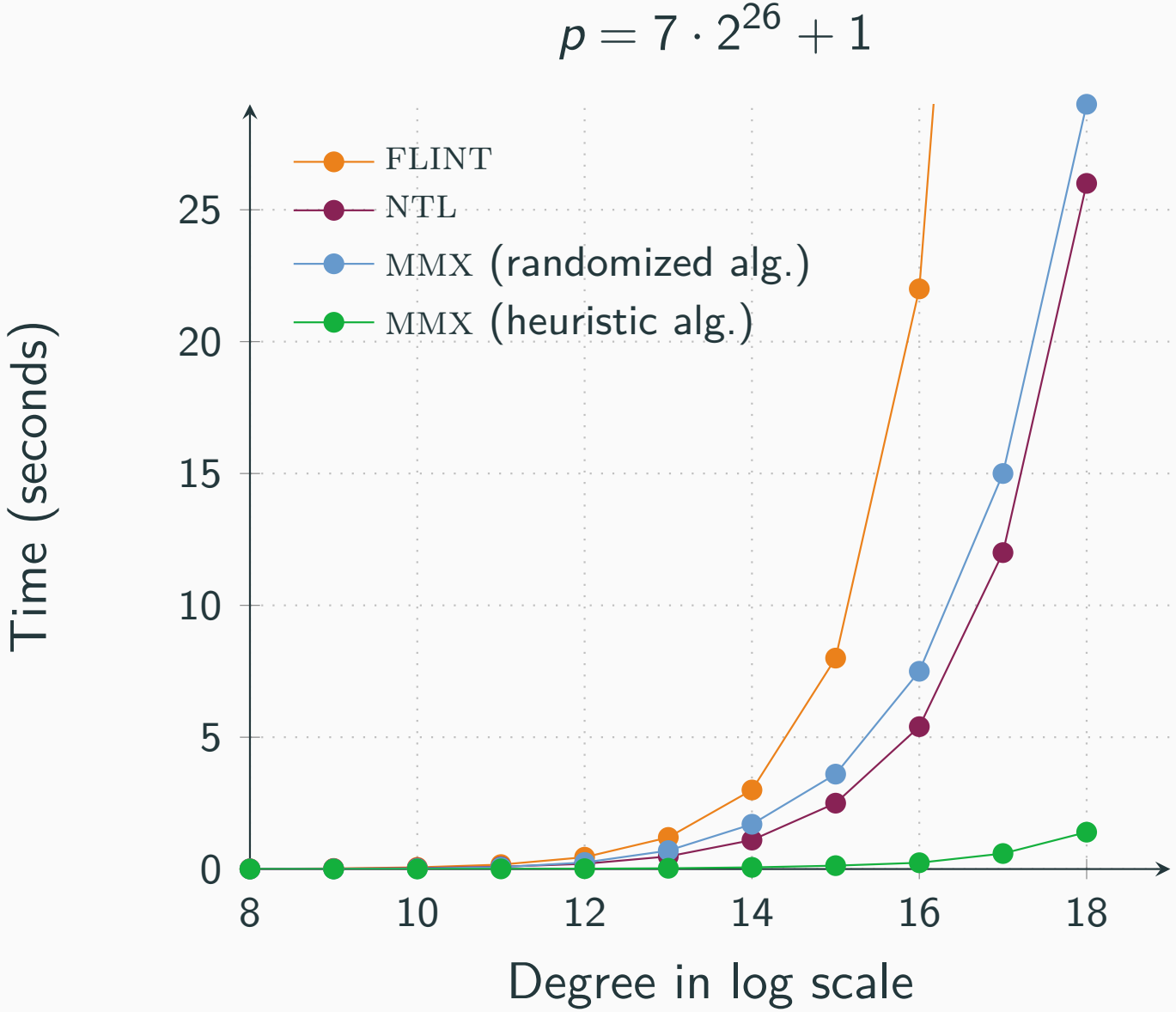


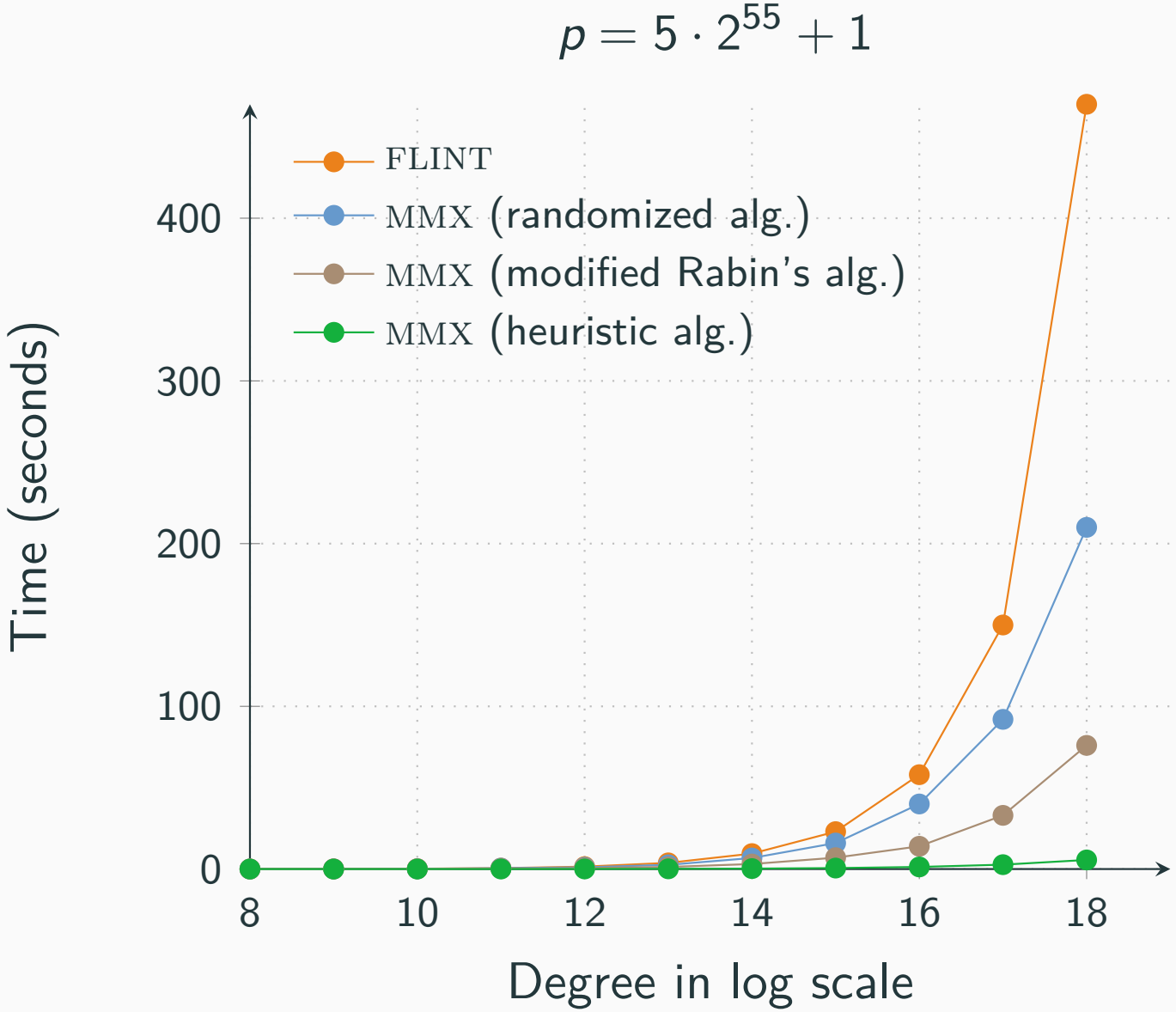
Theorem

Suppose that f is chosen at random in $\mathbb{F}_p[X]$ or that the heuristic holds. Given a primitive element of $\mathbb{F}_p^*$, the heuristic algorithm runs in **expected time** $\tilde{O}(d \log^2 p)$, for $p = M \cdot 2^m + 1$ with $M = O(\log p)$.

Running times


$p = 7 \cdot 2^{26} + 1$
FLINT
NTL
MMX (randomized alg.)
MMX (heuristic alg.)
Time (seconds)
0
5
10
15
20
25
8
10
12
14
16
18
Degree in log scale

Timings


$p = 5 \cdot 2^{55} + 1$
FLINT
MMX (randomized alg.)
MMX (modified Rabin's alg.)
MMX (heuristic alg.)
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0
100
200
300
400
8
10
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14
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Merci de votre attention !