Memory-efficient polynomial arithmetic

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Polynomial arithmetic

- Multiplication: $\mathrm{M}(n)$
  - Naïve: $2n^2 + 2n - 1$
  - Karatsuba: $< 6.5n^{\log_2 3}$
  - Toom-3: $< 18.75n^{\log_3 5}$
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  - GCD: $O(\mathrm{M}(n) \log n)$
  - Evaluation & interpolation: $O(\mathrm{M}(n) \log n)$
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What about space complexity?

Space complexity of polynomial arithmetic

- Quadratic multiplication algorithm: $O(1)$ [1]
- Karatsuba, Toom-3, FFT: $O(n)$
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- Improvements on FFT-based algorithms:
  - Roche (2009): $O(1)$ if $n = 2^k$
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  - Divide-and-Conquer: $\left(f_0 + X^{\frac{n}{2}} f_1\right) \cdot \left(g_0 + X^{\frac{n}{2}} g_1\right)$
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  - $\rightsquigarrow$ $\mathrm{size}((F(\omega^i), G(\omega^i))_i) = 2\,\mathrm{size}(FG)$
  - Roche'09: Compute half of the result + recurse
  - Harvey-Roche'10: same with TFT (vdH'04)

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Results:

- Yes!
- Almost (for other products)

Outline

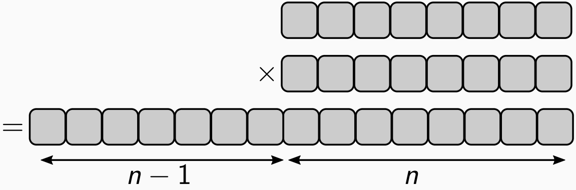
Polynomial products and linear maps

Space-preserving reductions

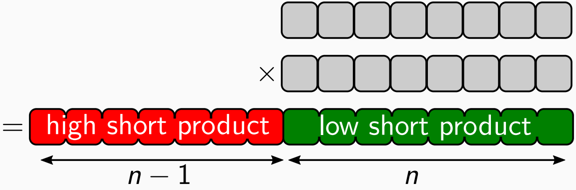
In-place algorithms from out-of-place algorithms

Polynomial products and linear maps

Short product



Short product



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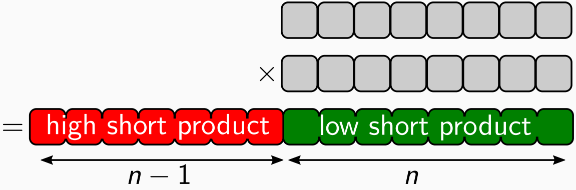
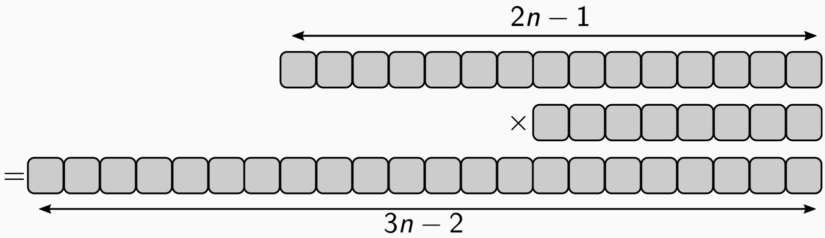


- Low short product: product of truncated power series
- Useful in other algorithms
- Time complexity: $\mathrm{M}(n)$
- Space complexity: $O(n)$

Middle product



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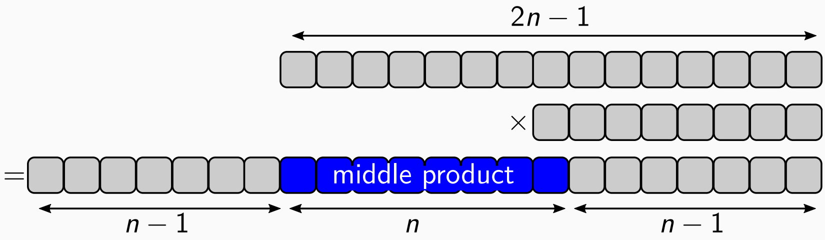

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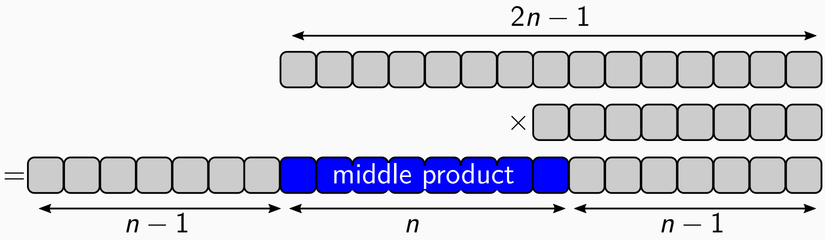
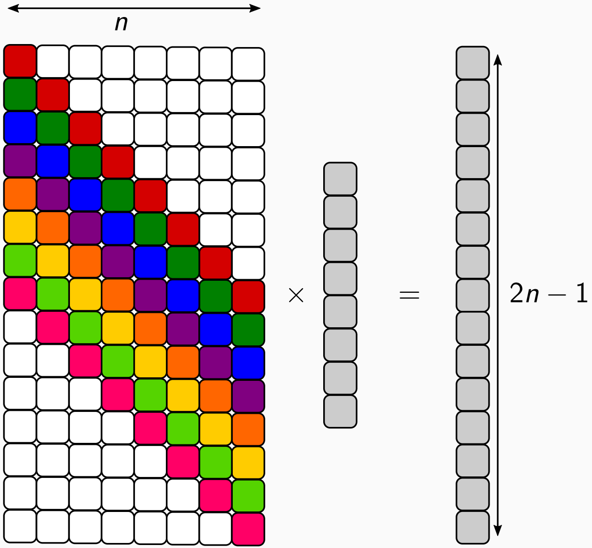


- Useful for Newton iteration
  - $G \leftarrow G(1-GF) \bmod X^{2n}$ with $GF = 1 + X^n H$
  - division, square root, ...
- Time complexity: $\mathsf{M}(n) \rightarrow$ Tellegen's transposition
- Space complexity: $O(n)$
- $O(1)$ space in the most permissive model via transposition of Harvey-Roche algorithm (Bostan-Lecerf-Schost'03)

Multiplications as linear maps



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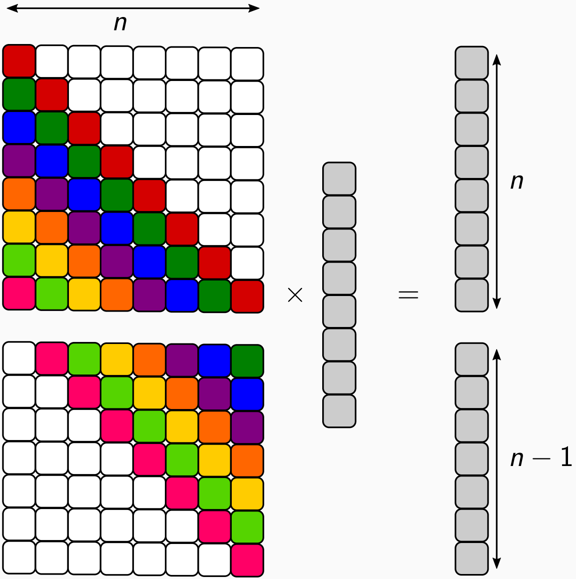

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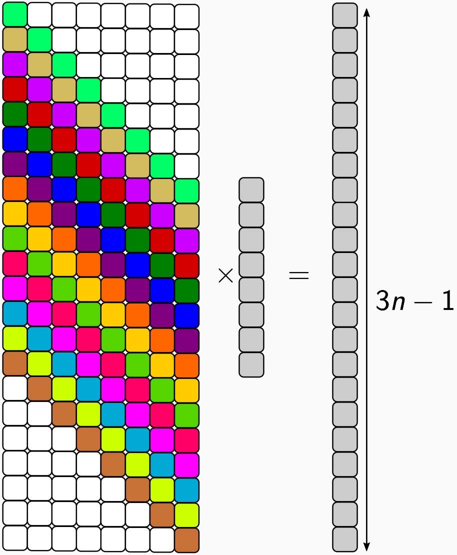

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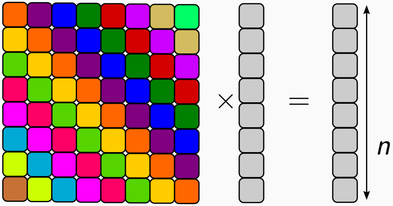

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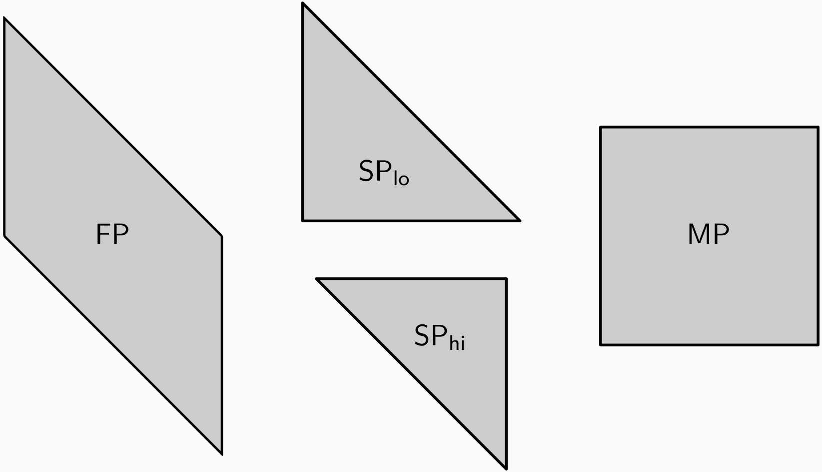

Full product | Short products | Middle product

Space-preserving reductions

Relative difficulties of products

- Without space restrictions:
    - $\mathsf{SP} \leq \mathsf{FP}$ and $\mathsf{FP} \leq \mathsf{SP}_{\mathsf{lo}} + \mathsf{SP}_{\mathsf{hi}}$
    - $\mathsf{MP} \equiv \mathsf{FP}$ (transposition)
    - $\mathsf{MP} \leq \mathsf{SP}_{\mathsf{lo}} + \mathsf{SP}_{\mathsf{hi}} + (n-1)$ additions

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- Size of inputs and outputs:
  - FP : $n \times n \to 2n-1$
  - $\mathrm{SP}_{\mathrm{lo}}$: $n \times n \to n$
  - $\mathrm{SP}_{\mathrm{hi}}$: $n-1 \times n-1 \to n-1$;
  - MP : $2n-1 \times n \to n$

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Reductions unusable in space-restricted settings!

A relevant notion of reduction

Definitions

- $\mathrm{TISP}(t(n), s(n))$: computable in time $t(n)$ and space $s(n)$
- $A \leq_c B$: A computable with oracle B and
  - constant number c of calls to oracle
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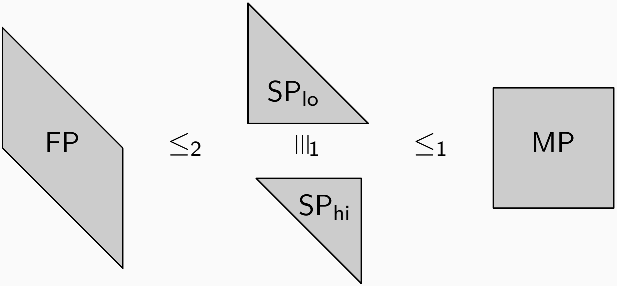
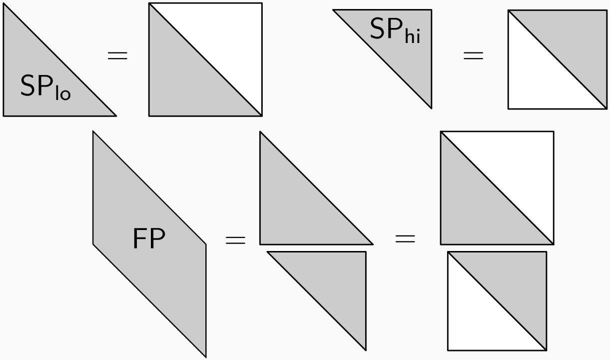
Proposition
If $B \in \mathrm{TISP}(t(n), s(n))$ and $A \leq_c B$, then

$$A \in \mathrm{TISP}(c\,t(n) + o(t(n)), s(n) + O(1))$$

Theorem



Visual proof



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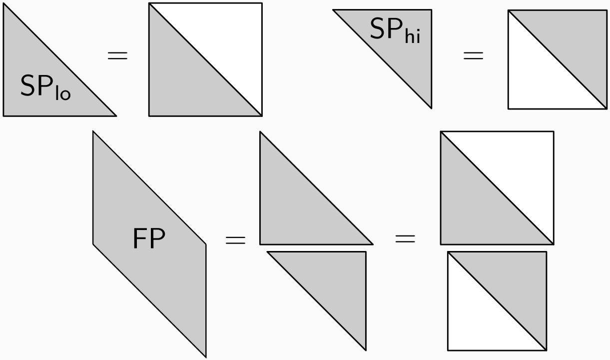

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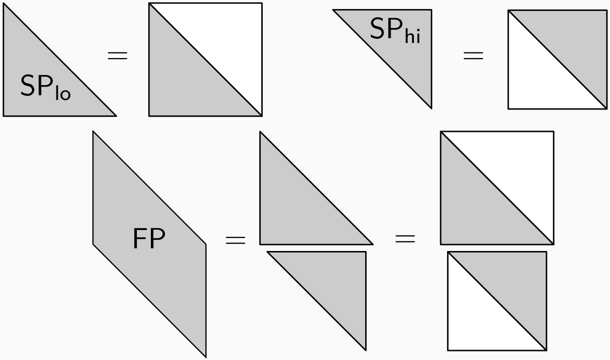

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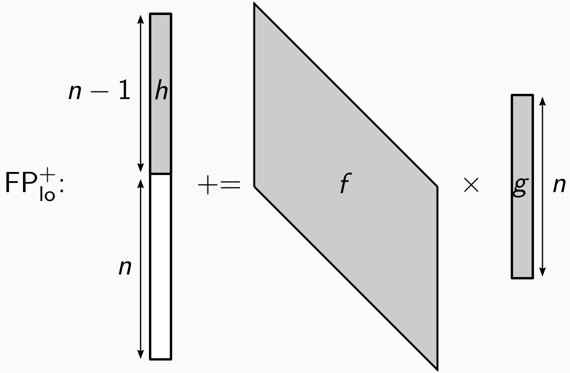
- $\mathrm{SP_{lo}}(n) \leq \mathrm{MP}(n)$; $\mathrm{SP_{hi}}(n) \leq \mathrm{MP}(n-1)$
- $\mathrm{FP}(n) \leq \mathrm{SP_{hi}}(n) + \mathrm{SP_{lo}}(n) \leq \mathrm{MP}(n) + \mathrm{MP}(n-1)$

Half-additive full product: $h \leftarrow h + f \cdot g$



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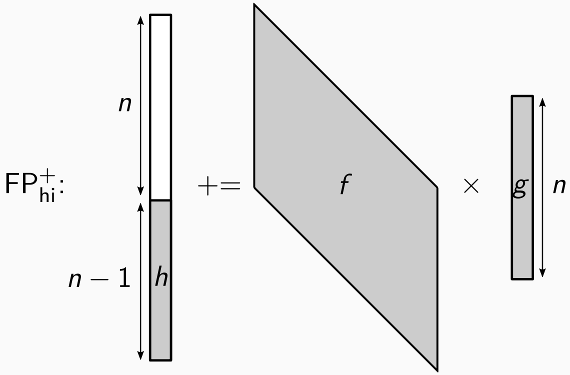

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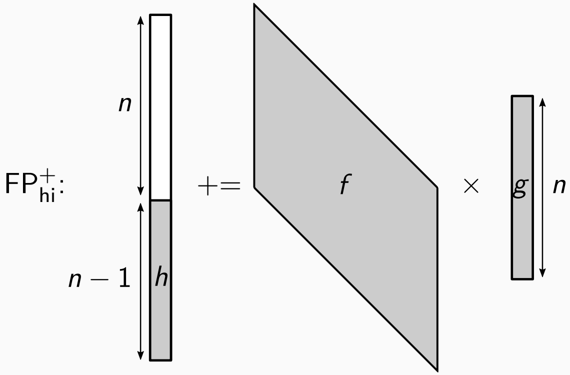
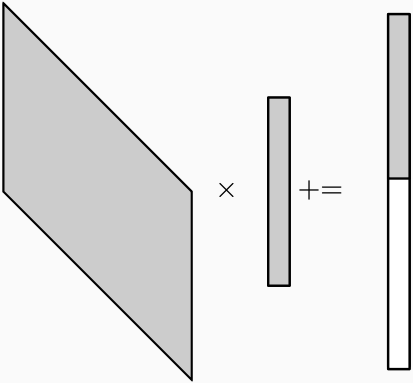


Remark $\mathrm{FP}^+_{\mathrm{lo}} \equiv_1 \mathrm{FP}^+_{\mathrm{hi}}$

Theorem $\mathrm{FP}^+ \leq_{3/2} \mathrm{SP}$ and $\mathrm{SP} \leq_2 \mathrm{FP}^+$

From SP to FP$^+$



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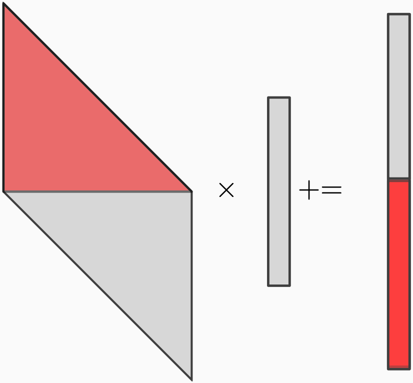

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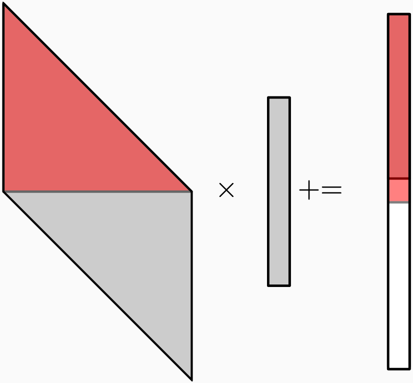

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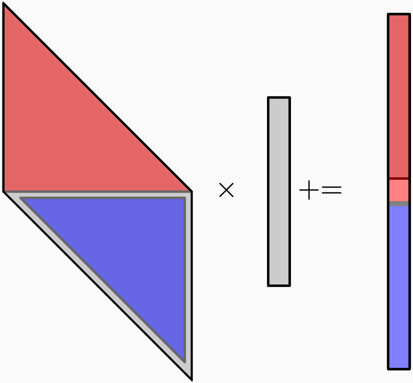

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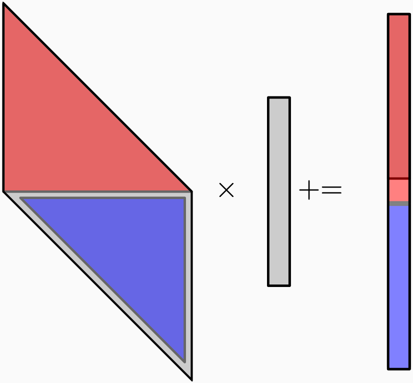

$$\mathsf{FP}^+_{\mathsf{lo}}(n) \leq \mathsf{SP}_{\mathsf{lo}}(n) + \mathsf{SP}_{\mathsf{hi}}(n) + n - 1$$

From FP$^+$ to SP

$$\left(f_0 + X^{\lceil n/2 \rceil} f_1\right) \cdot \left(g_0 + X^{\lceil n/2 \rceil} g_1\right) = f_0 g_0 + X^{\lceil n/2 \rceil}(f_0 g_1 + f_1 g_0) \mod X^n$$

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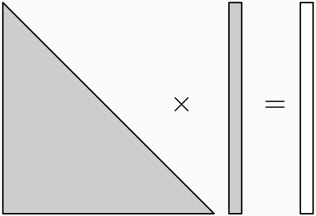

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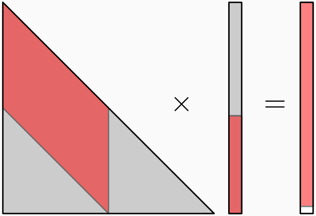

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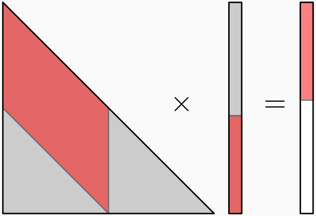

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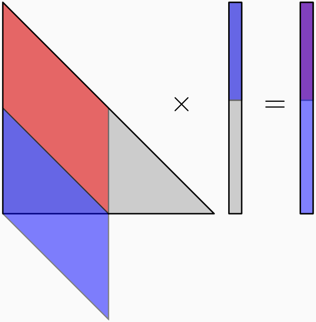

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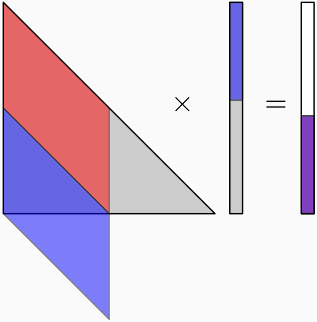

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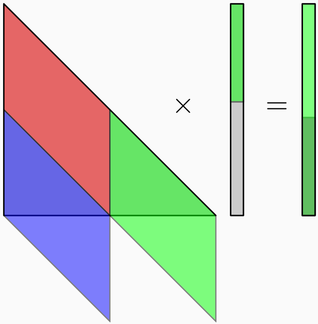
$$\left(f_0 + X^{\lceil n/2 \rceil} f_1\right) \cdot \left(g_0 + X^{\lceil n/2 \rceil} g_1\right) = f_0 g_0 + X^{\lceil n/2 \rceil} (f_0 g_1 + f_1 g_0) \mod X^n$$



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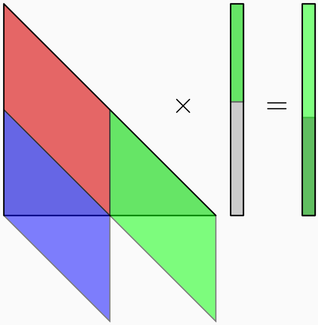

$$\mathrm{SP}_{\mathrm{lo}}(n) \le \mathrm{FP}(\lfloor n/2\rfloor) + \mathrm{FP}^+_{\mathrm{lo}}(\lfloor n/2\rfloor) + \mathrm{FP}^+_{\mathrm{hi}}(\lceil n/2\rceil)$$

Converse directions?

- From FP to SP:
    - problem with the output size
    - without space restriction: is $\mathsf{SP}(n) \simeq \mathsf{FP}(n/2)$?

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- From FP to SP:
  - problem with the output size
  - without space restriction: is $\mathrm{SP}(n) \simeq \mathrm{FP}(n/2)$?
- From SP to MP:
  - partial result:
    - up to $\log(n)$ increase in time complexity
    - techniques from next part
  - without space restriction or in a permissive model
    - FP to MP through Tellegen's transposition principle

In-place algorithms from out-of-place algorithms

Framework

- In-place algorithms parametrized by out-of-place algorithm
  - Out-of-place: Uses cn extra space
  - Constant c known to the algorithm

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  - Out-of-place: Uses cn extra space
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  - Space complexity: $O(1)$
  - Time complexity: closest to the out-of-place algorithm
- Technique:
  - Oracle calls in smaller size
  - Tail recursive call
  - *Fake* padding

Tail recursion and fake padding

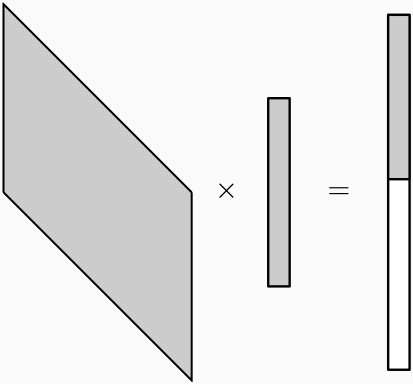
- Tail recursion:
  - Only one recursive call + last (or first) instruction
  - No need of recursive stack $\rightsquigarrow$ avoid $O(\log n)$ extra space

Tail recursion and fake padding

- Tail recursion:
  - Only one recursive call + last (or first) instruction
  - No need of recursive stack $\rightsquigarrow$ avoid $O(\log n)$ extra space
- *Fake* padding:
  - Pretend to pad inputs with zeroes
  - Make the data structure responsible for it
    - $O(1)$ increase in memory
    - *Cf.* strides in dense linear algebra
  - OK in inputs, not in outputs!

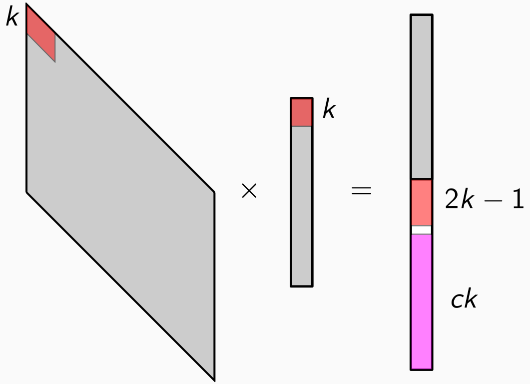
In-place FP$^+$ from out-of-place FP

$$(f_0 + X^k \hat{f}) \cdot (g_0 + X^k \hat{g}) = f_0 g_0 + X^k (f_0 \hat{g} + \hat{f} g_0) + X^{2k} \hat{f} \hat{g}$$



In-place FP$^+$ from out-of-place FP

$$(f_0 + X^k \hat{f}) \cdot (g_0 + X^k \hat{g}) = {\color{red} f_0 g_0} + X^k(f_0 \hat{g} + \hat{f} g_0) + X^{2k} \hat{f} \hat{g}$$



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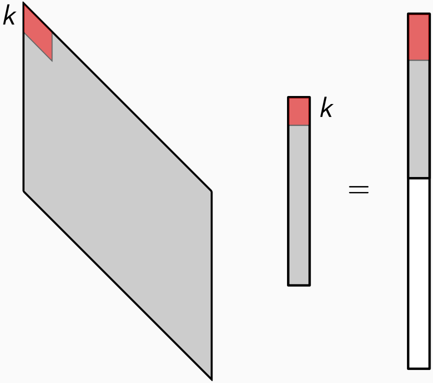

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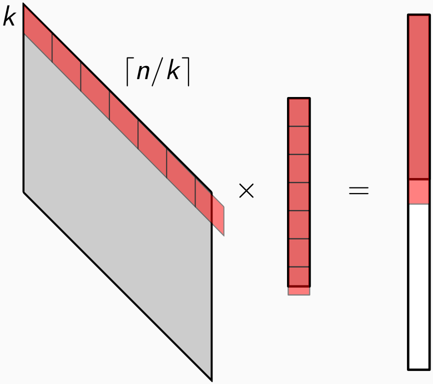

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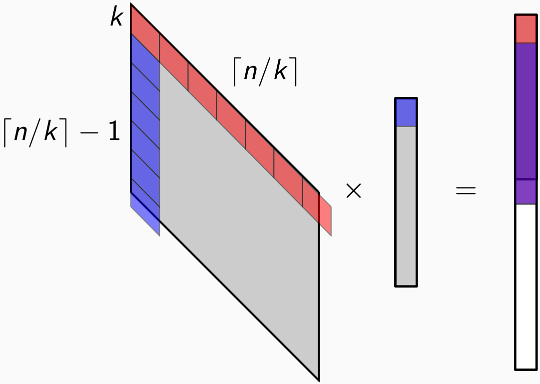

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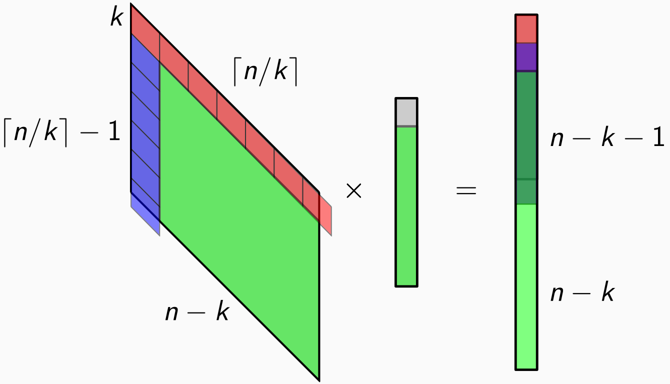
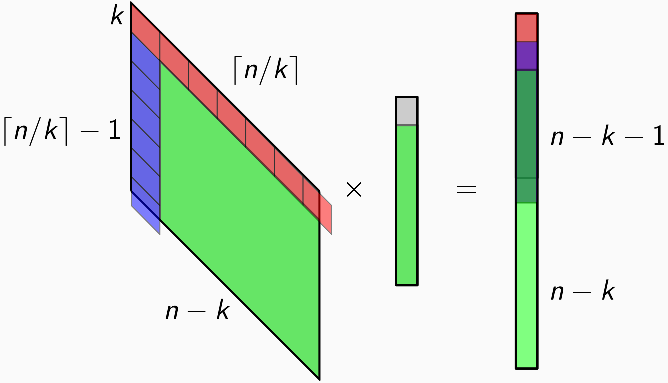


Analysis



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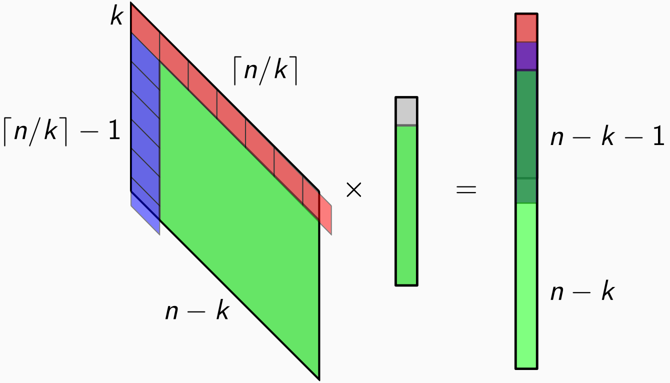

- $ck + 2k - 1 \leq n - k \implies k \leq \frac{n+1}{c+3}$
- $T(n) = (2\lceil n/k \rceil - 1)(\mathrm{M}(k) + 2k - 1) + T(n-k)$

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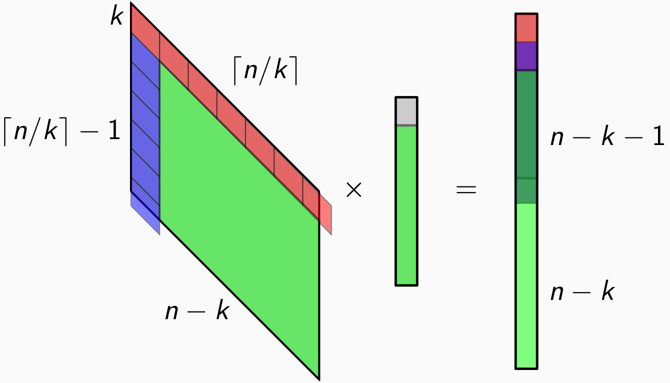

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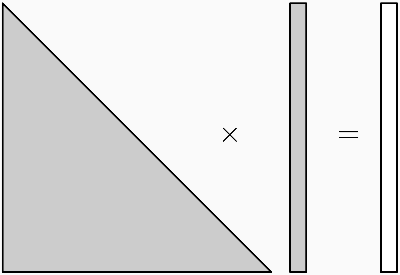
In-place short product



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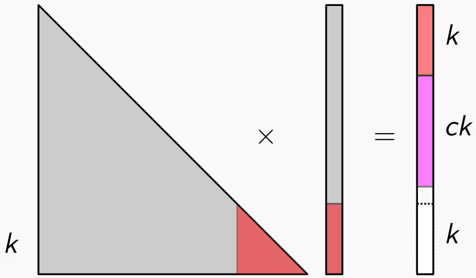

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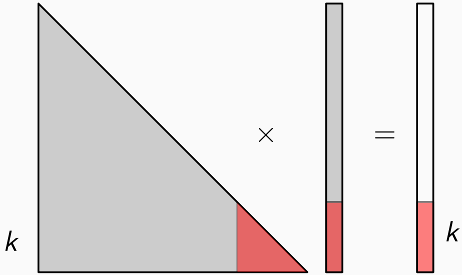

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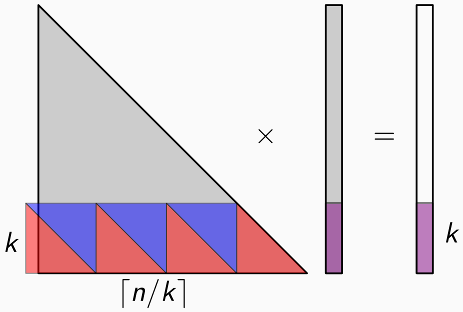

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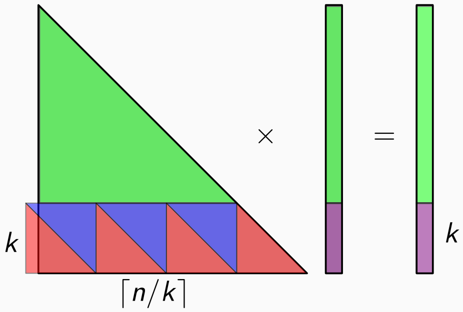

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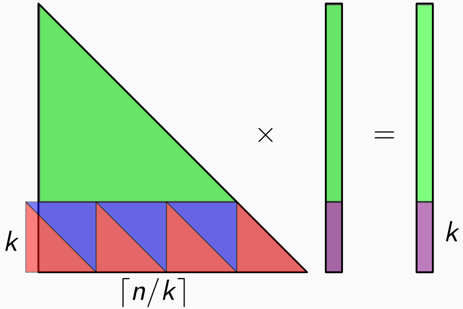

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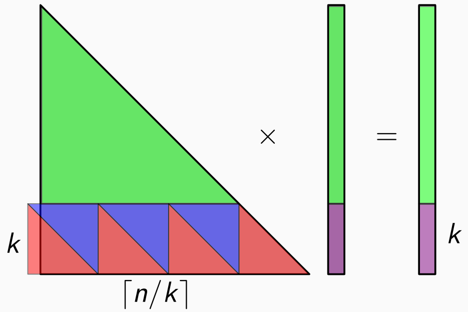

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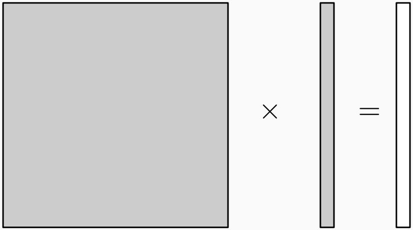

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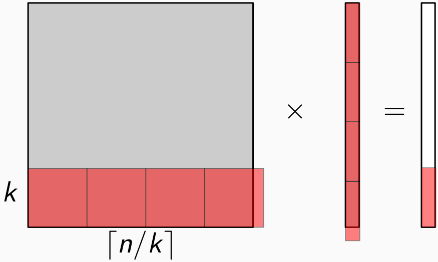

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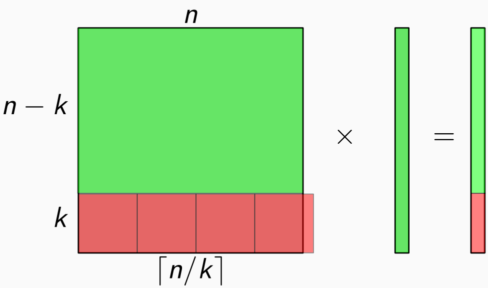

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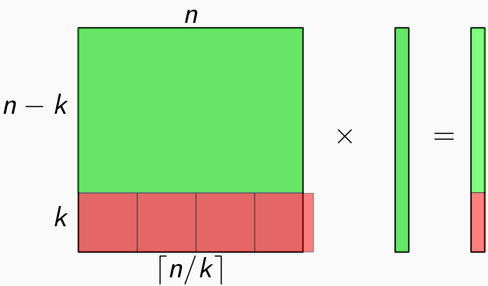

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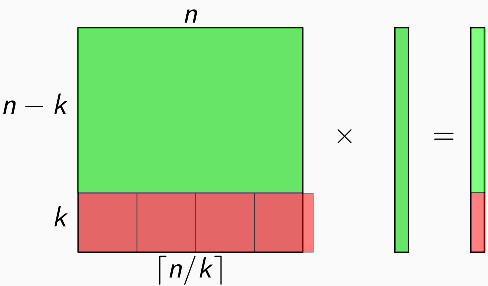

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$$T(n,n) \leq \begin{cases} \mathsf{M}(n) \log_{\frac{c+2}{c+1}}(n) + o(\mathsf{M}(n)\log n) & \text{if } \mathsf{M}(n) \text{ is quasi-linear} \\ O(\mathsf{M}(n)) & \text{otherwise} \end{cases}$$

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Work in progress!

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- Use our in-place algorithms as building blocks
  - Newton iteration: division, square root, ...
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$\rightarrow$ (at most) $\log(n)$ increase in complexity

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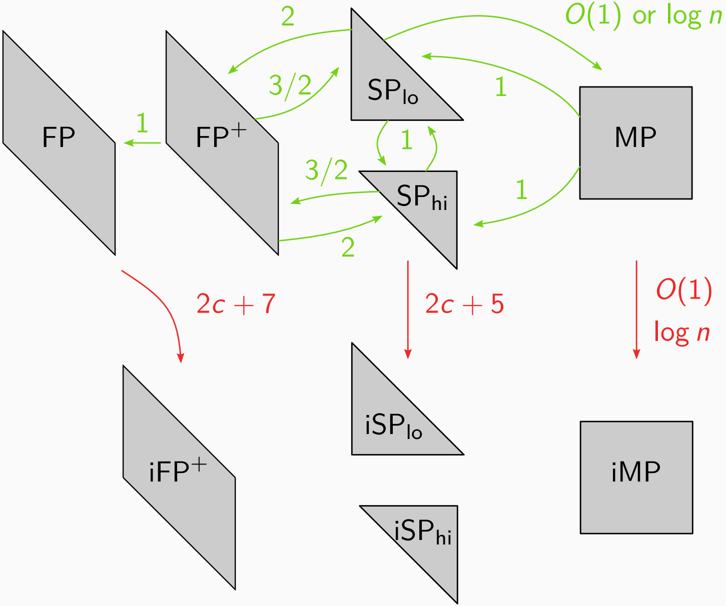
Remark

- In place: division with remainder
- Only quotient or only remainder: not clear
- Main difficulty: size of the output

Summary



2
$O(1)$ or $\log n$
SP_{lo}
$3/2$
1
FP
1
FP^{+}
1
MP
$3/2$
SP_{hi}
1
2
$2c+7$
$2c+5$
$O(1)$
$\log n$
iSP_{lo}
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Thank you!