Memory-efficient polynomial arithmetic

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Séminaire LACL — 26 nov. 2018

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Polynomial arithmetic

- Multiplication: $\mathrm{M}(n)$
  - Naïve: $2n^2 + 2n - 1$
  - Karatsuba: $< 6.5n^{\log_2 3}$
  - Toom-3: $< 18.75n^{\log_3 5}$
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  - GCD: $O(\mathsf{M}(n) \log n)$
  - Evaluation & interpolation: $O(\mathsf{M}(n) \log n)$
  - Power series computations: $O(\mathsf{M}(n))$ or $O(\mathsf{M}(n) \log n)$
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What about space complexity?

Space complexity of polynomial arithmetic

- Quadratic multiplication algorithm: $O(1)$ [1]
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- Improvements on FFT-based algorithms:
  - Roche (2009): $O(1)$ if $n = 2^k$
  - Harvey & Roche (2010): $O(1)$

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$\rightarrow$ *Standard* registers of size $O(\log n)$

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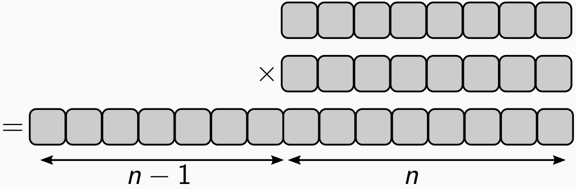
  - *Reasonable* from a programmer's viewpoint
- Read-write input and output
  - Too permissive in general
  - Special case: inputs must be restored at the end

Short product



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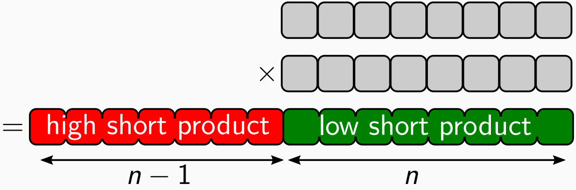

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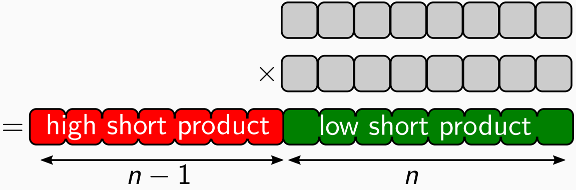
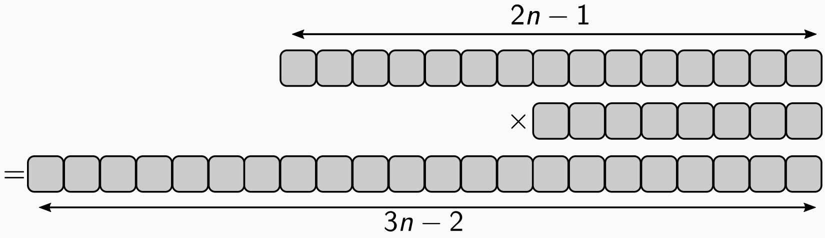


- Low short product: product of truncated power series
- Useful in other algorithms
- Time complexity: $\mathrm{M}(n)$
- Space complexity: $O(n)$

Middle product



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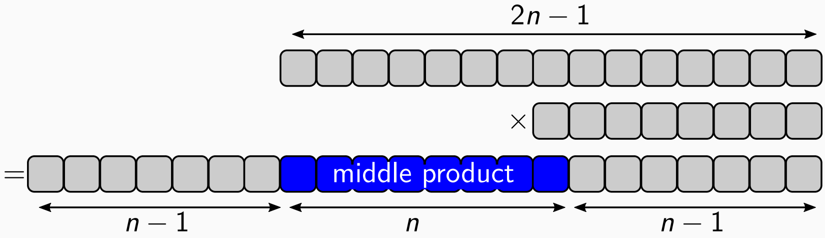

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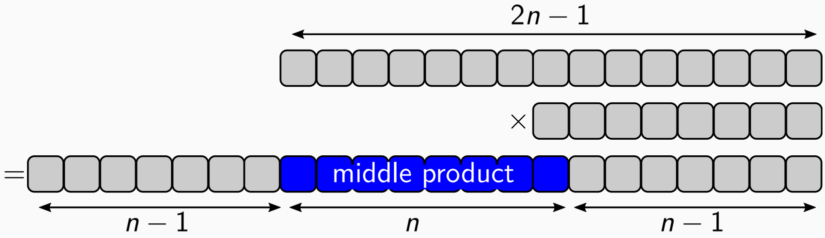
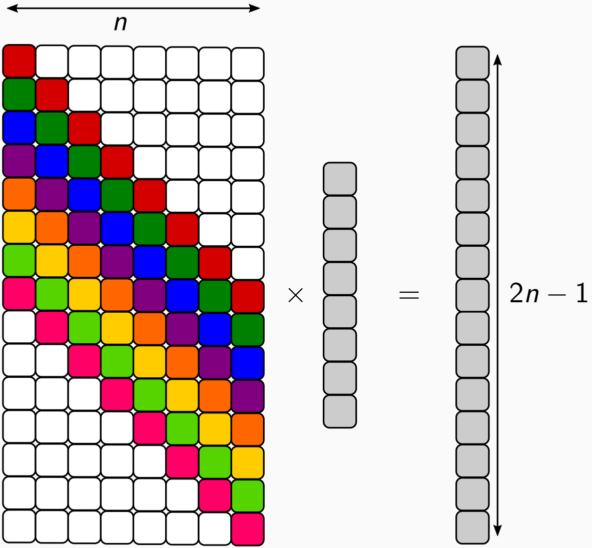


- Useful for Newton iteration
  - $G \leftarrow G(1 - GF) \bmod X^{2n}$ with $GF = 1 + X^n H$
  - division, square root, ...
- Time complexity: $\mathsf{M}(n) \rightarrow$ Tellegen's transposition
- Space complexity: $O(n)$

Multiplications as linear maps



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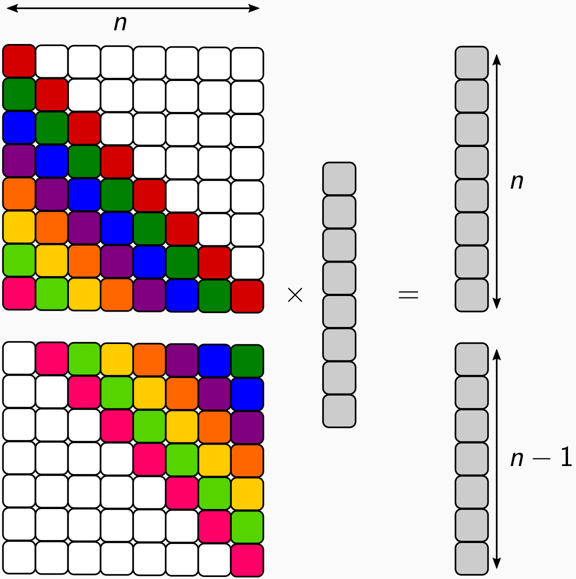

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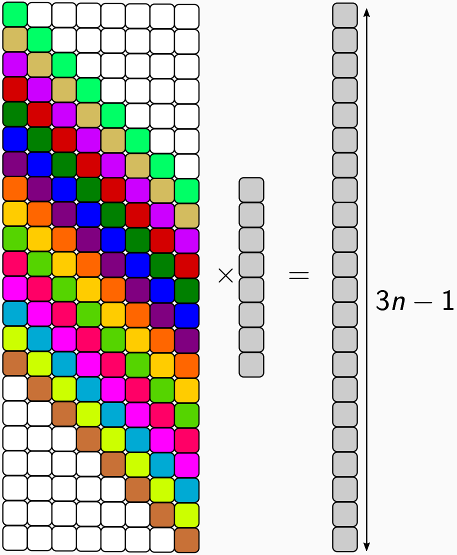

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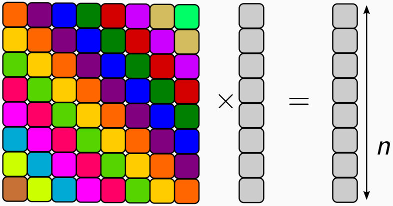

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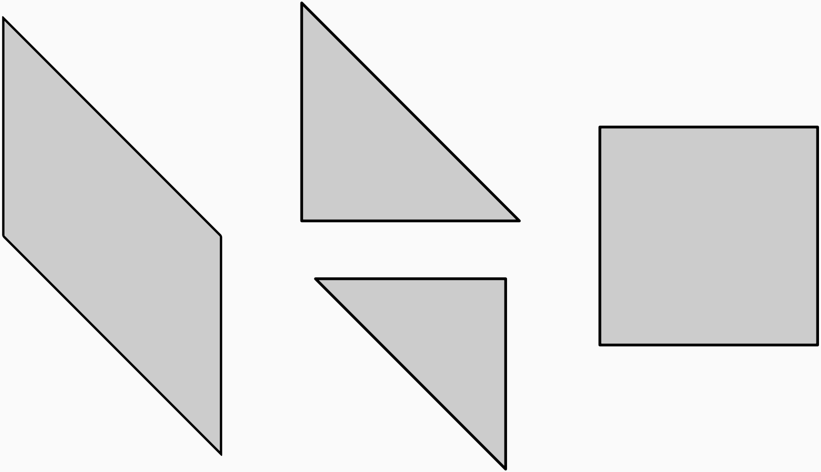

Full product | Short products | Middle product

Outline

Space-preserving reductions

In-place algorithms from out-of-place algorithms

Space-preserving reductions

Reductions

Definitions.

- $\text{TISP}(t(n), s(n))$: decidable in time $t(n)$ and space $s(n)$
- $A \leq B$: A decidable with oracle B
  - constant number of calls to oracle
  - negligible extra time
  - without extra space ($O(1)$)
- $A \equiv B$: $A \leq B$ and $B \leq A$

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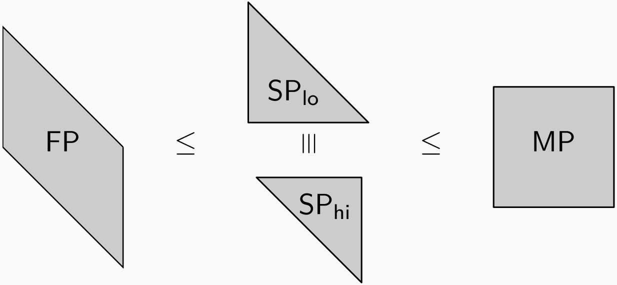
Proposition.
If $B \in \mathrm{TISP}(t(n), s(n))$ and $A \leq B$, then

$$A \in \mathrm{TISP}(O(t(n)), s(n) + O(1))$$

Results

Theorem.



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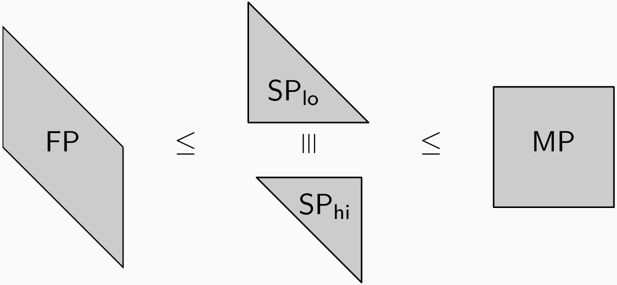
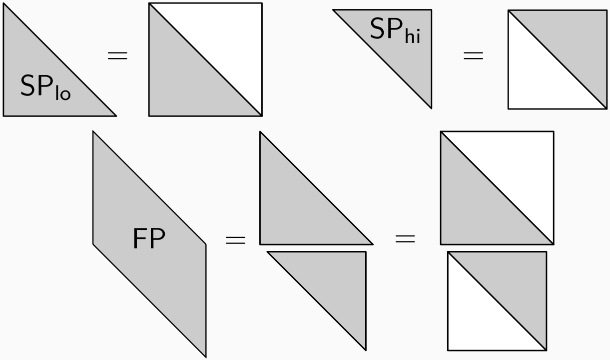


Remark.

- FP : $n \times n \to 2n - 1$
- SP_{lo} : $n \times n \to n$; SP_{hi} : $n - 1 \times n - 1 \to n - 1$;
- MP : $2n - 1 \times n \to n$

Visual proof



- Use of *fake padding* (in input, **not** in output!)

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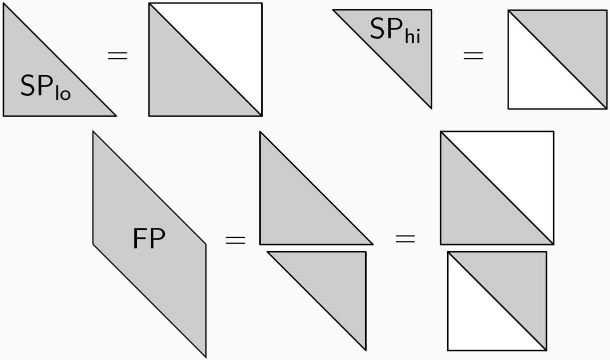

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- $SP_{lo}(n) \leq MP(n)$; $SP_{hi}(n) \leq MP(n-1)$

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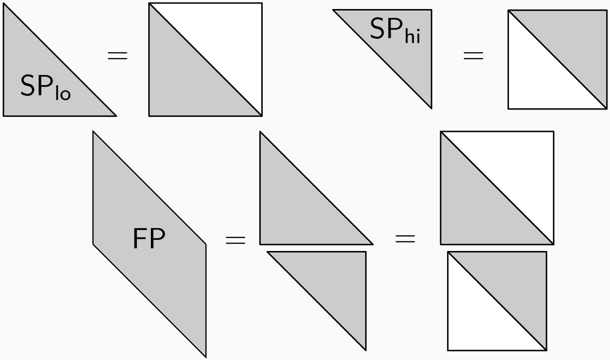
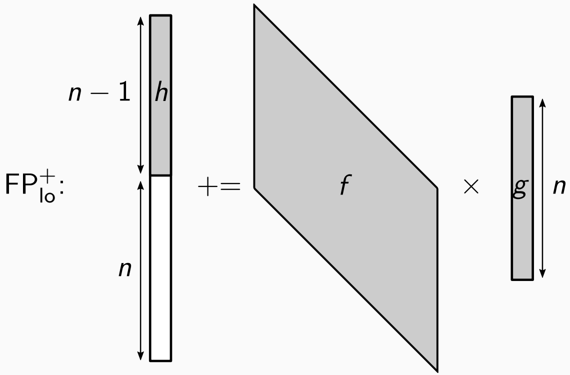


- Use of *fake padding* (in input, **not** in output!)
- $\mathrm{SP_{lo}}(n) \leq \mathrm{MP}(n)$; $\mathrm{SP_{hi}}(n) \leq \mathrm{MP}(n-1)$
- $\mathrm{FP}(n) \leq \mathrm{SP_{hi}}(n) + \mathrm{SP_{lo}}(n) \leq \mathrm{MP}(n) + \mathrm{MP}(n-1)$

Half-additive full product: $h \leftarrow h + f \cdot g$



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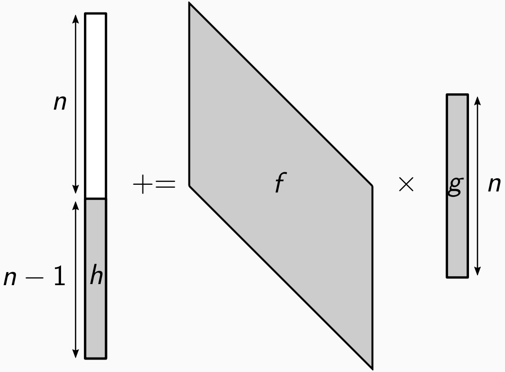
$\mathrm{FP}^{+}_{\mathrm{hi}}$:



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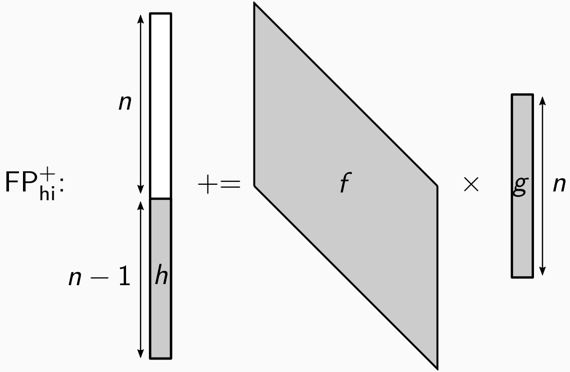
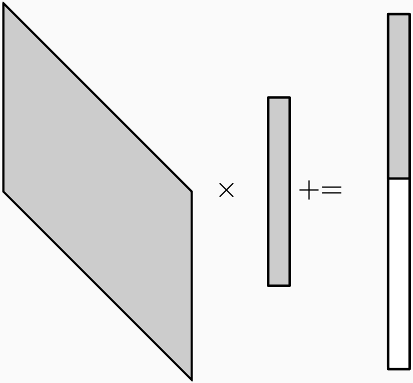


Remark. $\mathsf{FP}^+_{\mathsf{lo}} \equiv \mathsf{FP}^+_{\mathsf{hi}}$

Theorem. $\mathsf{FP}^+ \equiv \mathsf{SP}$

SP to FP^+



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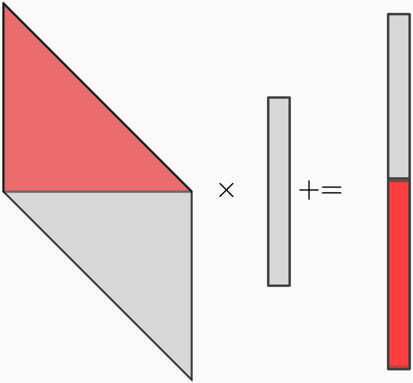

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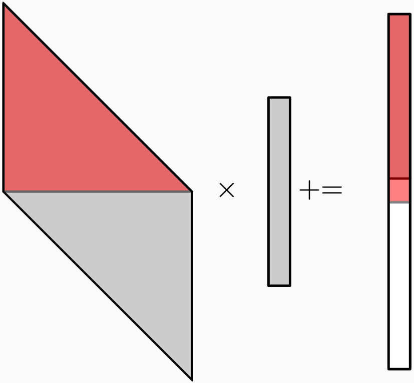

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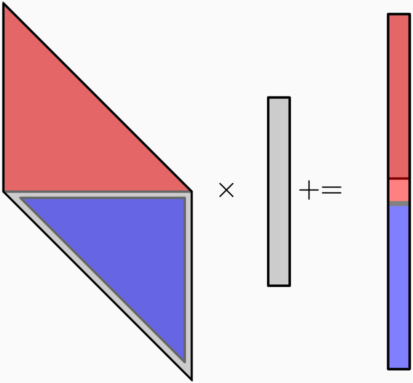

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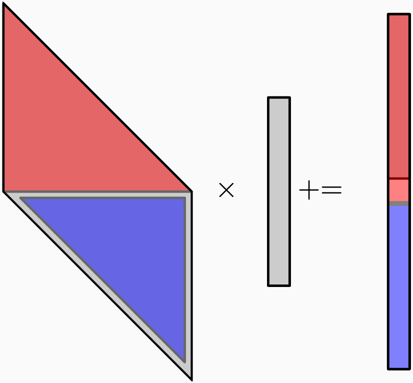

$$FP^+_{lo}(n) \leq SP_{lo}(n) + SP_{hi}(n) + n - 1$$

FP^+ to SP

$$\left(f_0 + X^{\lceil n/2 \rceil} f_1\right) \cdot \left(g_0 + X^{\lceil n/2 \rceil} g_1\right) = f_0 g_0 + X^{\lceil n/2 \rceil}(f_0 g_1 + f_1 g_0) \mod X^n$$

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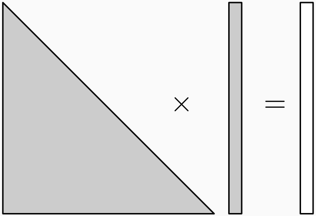

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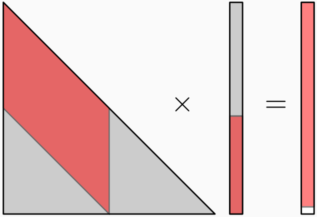

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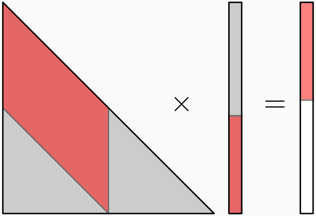

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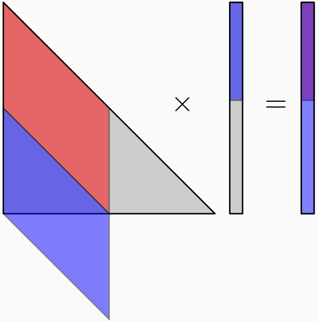

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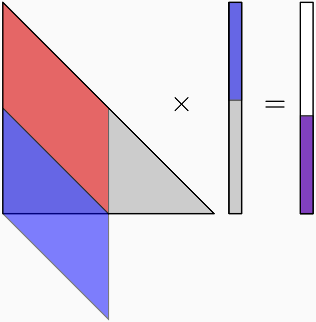

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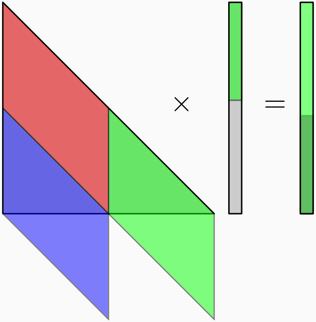

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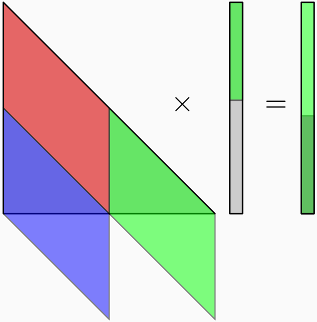

$$\mathrm{SP}_{\mathrm{lo}}(n) \leq \mathrm{FP}(\lfloor n/2 \rfloor) + \mathrm{FP}^+_{\mathrm{lo}}(\lfloor n/2 \rfloor) + \mathrm{FP}^+_{\mathrm{hi}}(\lceil n/2 \rceil)$$

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- From FP to SP:
  - problem with the output size
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- From FP to MP:
  - partial result: $\log(n)$ increase in time complexity
  - without space restriction: Tellegen's transposition principle

In-place algorithms from out-of-place algorithms

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- Technique:
  - Oracle calls in smaller size
  - Recursive call

In-place FP$^+$

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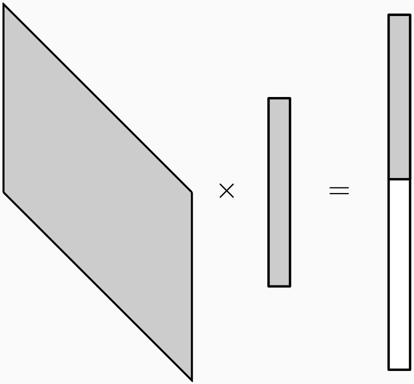

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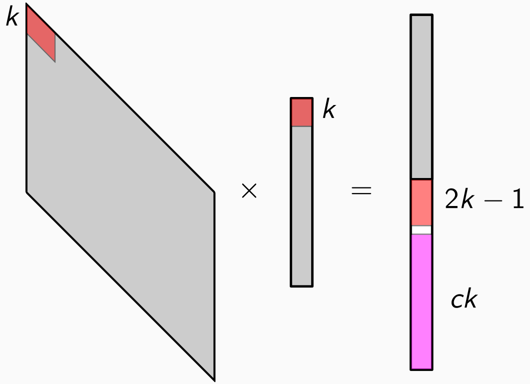

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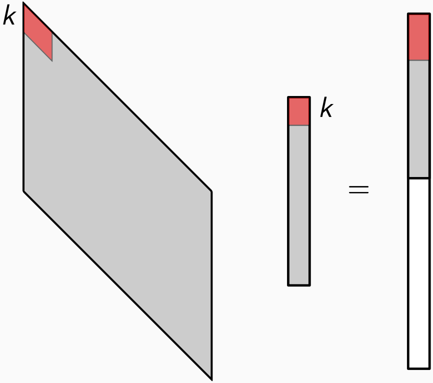

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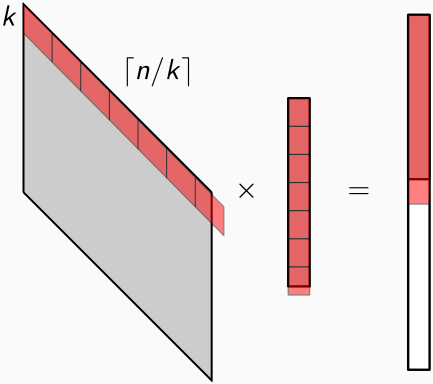

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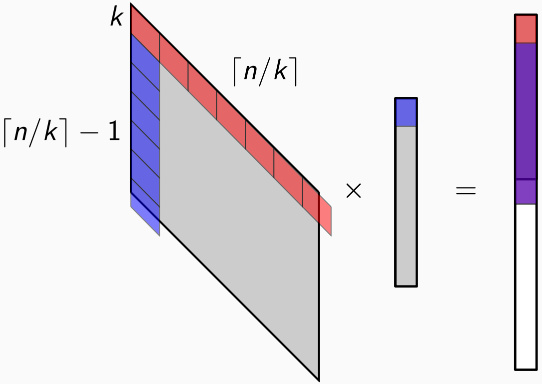

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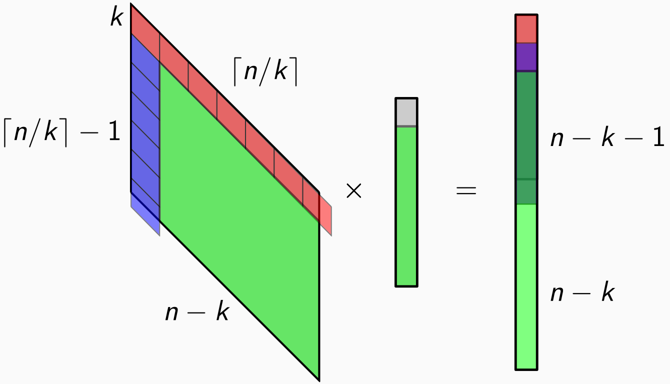

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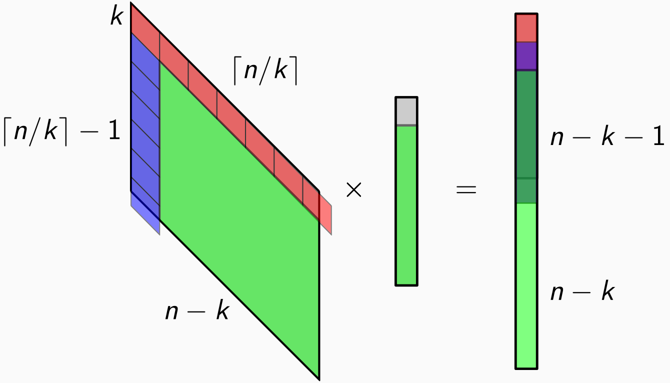

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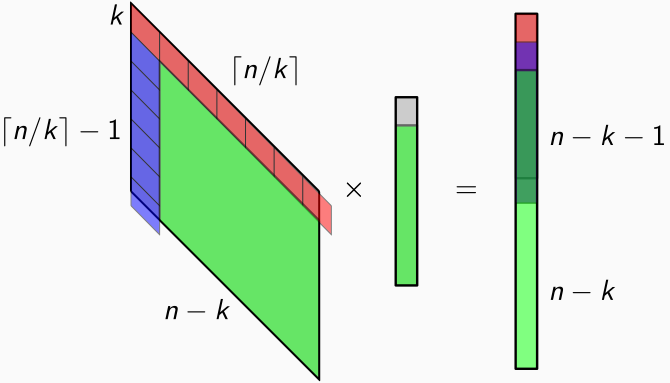

- $ck + 2k - 1 \leq n - k \rightarrow k \leq \frac{n+1}{c+3}$
- $T(n) = (2\lceil n/k \rceil - 1)(\mathrm{M}(k) + 2k - 1) + T(n-k)$

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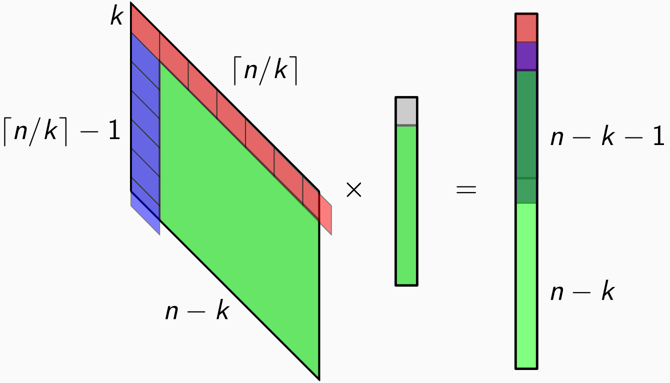

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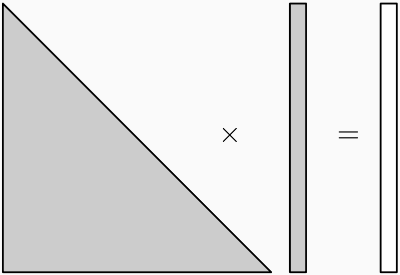
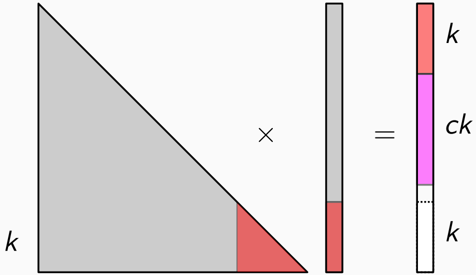
- $T(n) = (2\lceil n/k \rceil - 1)(\mathrm{M}(k) + 2k - 1) + T(n - k)$

$$T(n) \leq (2c + 7)\mathrm{M}(n) + o(\mathrm{M}(n))$$

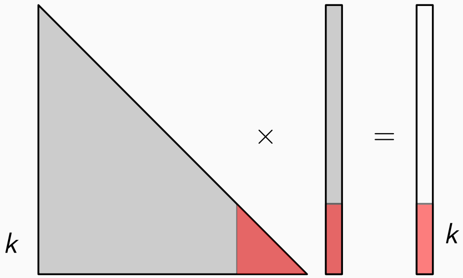
In-place short product



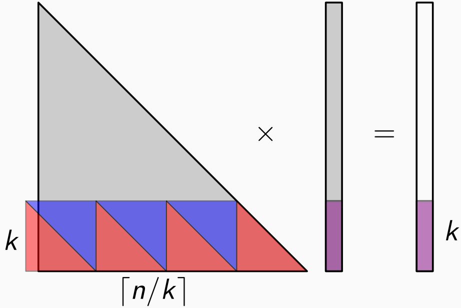
In-place short product



In-place short product



In-place short product



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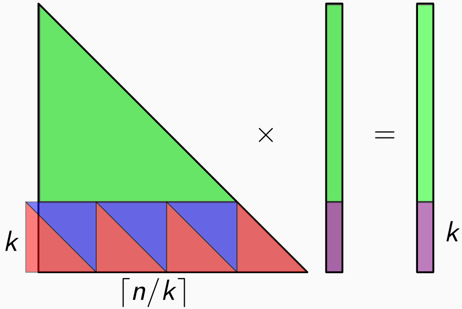

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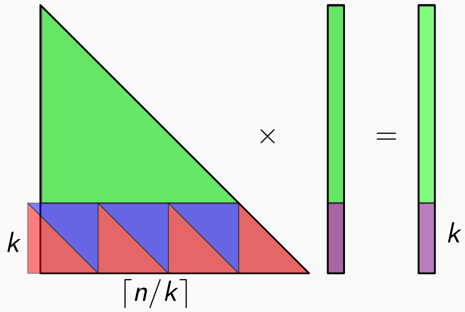

- $k \leq n/(c+2)$
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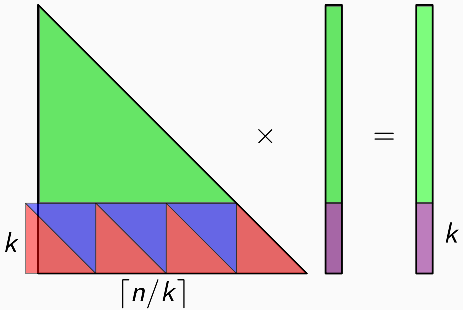

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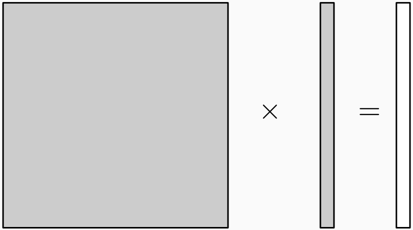
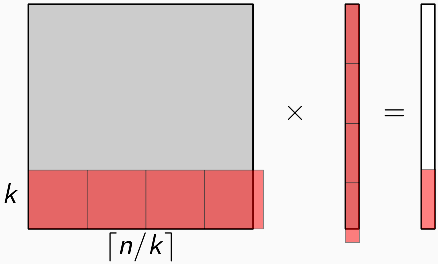
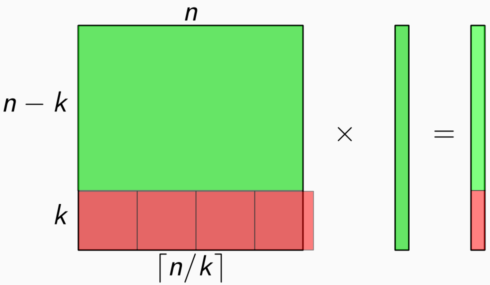
In-place middle product



In-place middle product



In-place middle product



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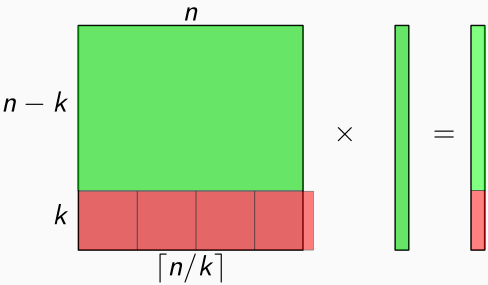

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- $T(n, m) = \lceil n/k \rceil \mathsf{M}(k) + T(n, m - k)$

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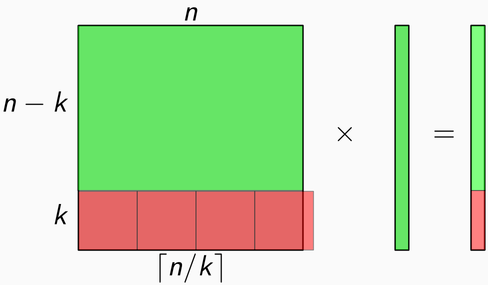

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$$T(n,m) \le \mathsf{M}(n)\log_{1+1/c+1}(m) + o(\mathsf{M}(n)\log m)$$

Other operations

Work in progress!

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- Use our in-place algorithms as building blocks
  - Newton iteration: division, square root, ...
  - Evaluation & interpolation

$\rightarrow$ (at most) $\log(n)$ increase in complexity

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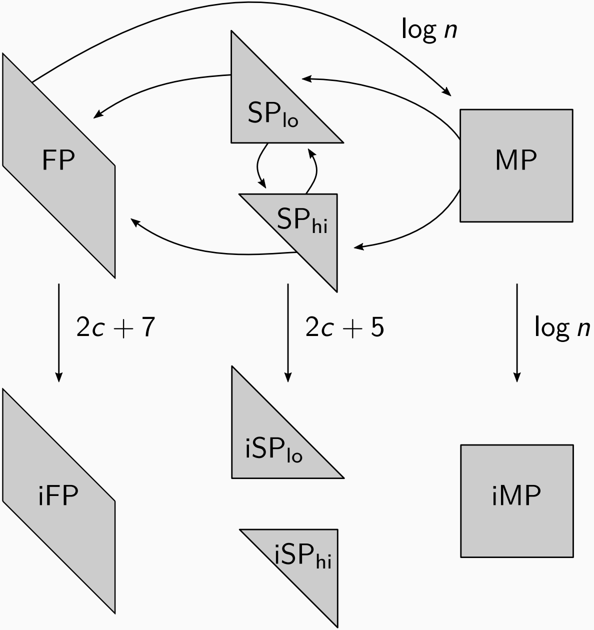
Remark.

- In place: division with remainder
- Only quotient or only remainder: not clear
- Main difficulty: size of the output

Summary



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Thank you!