Root finding over finite fields using Graeffe transforms

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Given $f \in \mathbb{F}_q[X]$, compute its roots, that is $\{\alpha \in \mathbb{F}_q : f(\alpha) = 0\}$.

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$$f(X) = \prod_{i=1}^{d} (X - \alpha_i), \quad \alpha_i \in \mathbb{F}_q^*, \quad \alpha_i \neq \alpha_j$$

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- Motivated by sparse interpolation **[van der Hoeven & Lecerf, 2014]**

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- Better complexity bounds when $q - 1$ is sufficiently *smooth*
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- **FFT finite field**: $p = M \cdot 2^m + 1$ with $M = O(\log p)$
  - Useful in practice
  - Adapt old algorithms
  - New technique based on **Graeffe transforms**
  - Fast implementations

Rabin's algorithm

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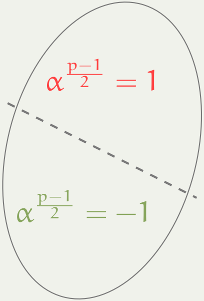

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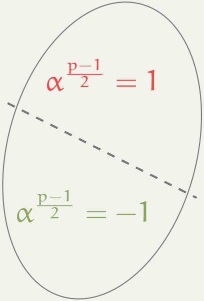

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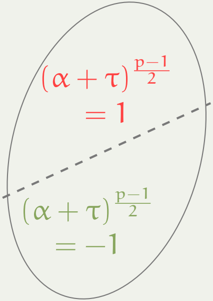
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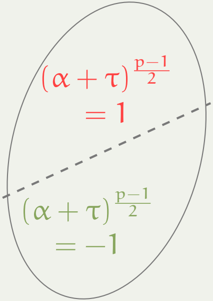

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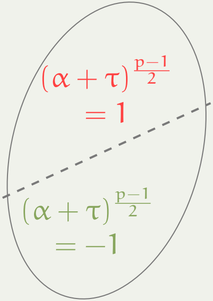

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The roots of $f \in \mathbb{F}_p[X]$ can be computed in expected time $\tilde{O}(d \log^2 p)$.

Modified Rabin's algorithm

(for FFT finite fields)

$$X^{p-1}-1=\prod_{i=0}^{2^\ell-1}(X^{M2^{m-\ell}}-\xi^i),\text{ where } \xi \text{ is primitive of order } 2^\ell.$$

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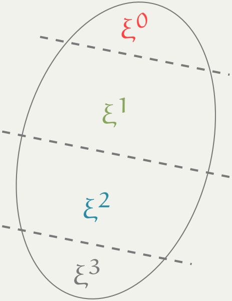

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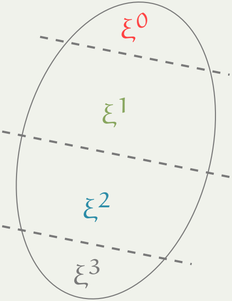

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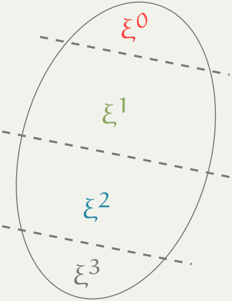

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Worthwhile in practice for **small** $\ell = 2, 3, \ldots$

The Graeffe transform

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$$f(X)f(-X) = \prod_i (X - \alpha_i)(-X - \alpha_i) = (-1)^d \prod_i (X^2 - \alpha_i^2)$$

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Remarks:

- $G_{\rho_1\rho_2} = G_{\rho_1} \circ G_{\rho_2}$, and in particular $G_{2^\ell} = G_2 \circ \cdots \circ G_2$
- $G_{p-1}(f)(X) = \prod_i (X - \alpha_i^{p-1}) = (X - 1)^d$

Using Graeffe transforms

$$f \xrightarrow{G_2} g_1 \xrightarrow{G_2} g_2 \xrightarrow{G_2} \cdots \xrightarrow{G_2} g_m \xrightarrow{G_M} g_{m+1}$$

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  - If $\beta = \zeta^e$, $\alpha_i, \alpha_j \in \{\zeta^{e/2}, \zeta^{(e+2^m M)/2}\}$

Deterministic complexity

Improvements and generalization:

- Modular composition for Graeffe transforms **[Kedlaya-Umans (2008)]**
- Fast discrete logarithms in $\mathbb{F}_q^*$ **[Pohlig-Hellman (1978)]**
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Theorem

Given $f \in \mathbb{F}_q[X]$ satisfying $(\mathcal{A})$, the irreducible factorization of $(q-1)$ and a primitive element of $\mathbb{F}_q^*$, the roots of f can be computed in time

$$\tilde{O}(\sqrt{S_1(q-1)}d\log^2 q) + (d\log^2 q)^{1+o(1)}$$

where $S_1(q-1)$ is the largest factor of $q-1$.

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- Refines Shoup's complexity bounds
- Note: If $q = M \cdot 2^m + 1$, $M = O(\log q)$, complexity $\tilde{O}(d\log^2 q)$.

Tangent Graeffe transform

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Lemma

Let $g + \varepsilon \overline{g} = G_{2^\ell}(f + \varepsilon f')$. A nonzero root β of g is **simple iff** $\overline{g}(\beta) \neq 0$. The corresponding root of f is $\alpha = 2^\ell \beta g'(\beta)/\overline{g}(\beta)$.

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Goal: Ensure many **simple roots**.

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- Same asymptotic as Rabin's algorithm
- Better efficiency in practice
- Primitive elements easy to compute in practice

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If $2^\ell \simeq p/d$, $G_{2^\ell}(f(X+\tau))$ has $\Omega(d)$ **simple roots** with probability $\geqslant 1/2$, for a random $\tau \in \mathbb{F}_p$.

Justification: holds for a random f rather than $f(X+\tau)$.

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If $2^\ell \simeq p/d$, $G_{2^\ell}(f(X+\tau))$ has $\Omega(d)$ **simple roots** with probability $\geqslant 1/2$, for a random $\tau \in \mathbb{F}_p$.

Justification: holds for a random f rather than $f(X+\tau)$.

$$\begin{tikzcd}
f(X+\tau+\varepsilon) \arrow[r, "G_{2^\ell}"] & g_\ell + \varepsilon\overline{g}_\ell \arrow[d, mapsto] \\
 & Z_\ell \\
 & \cap \\
 & \{\xi^e : 0 \leqslant e < M \cdot 2^\ell\}
\end{tikzcd}$$

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Z_0 & & Z_\ell \arrow[ll, "\text{only simple roots}"'] \arrow[d, phantom, "\cap"] \\
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Z_0 \arrow[u, bend left, "\begin{array}{c}\text{recursive call:}\\ f/\prod_{\alpha\in Z_0}(X-\alpha)\end{array}"] & Z_\ell \arrow[l, "\text{only simple roots}"'] \arrow[d, phantom, "\cap"] \\
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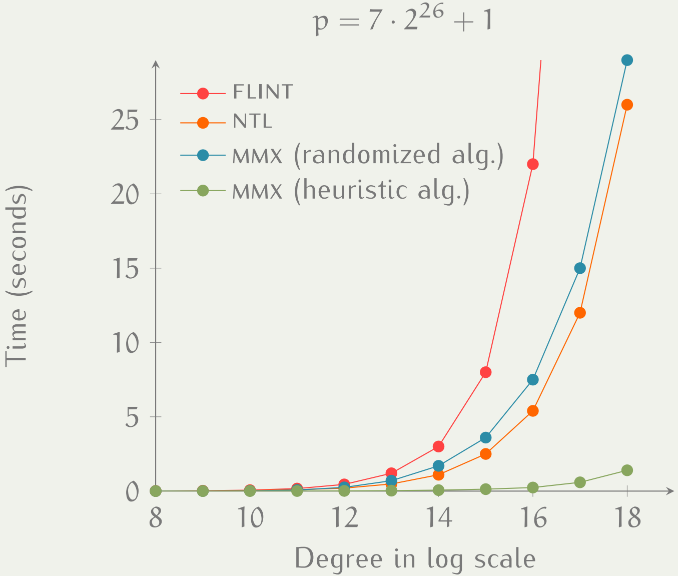
\end{tikzcd}$$

Heuristic complexity

Theorem

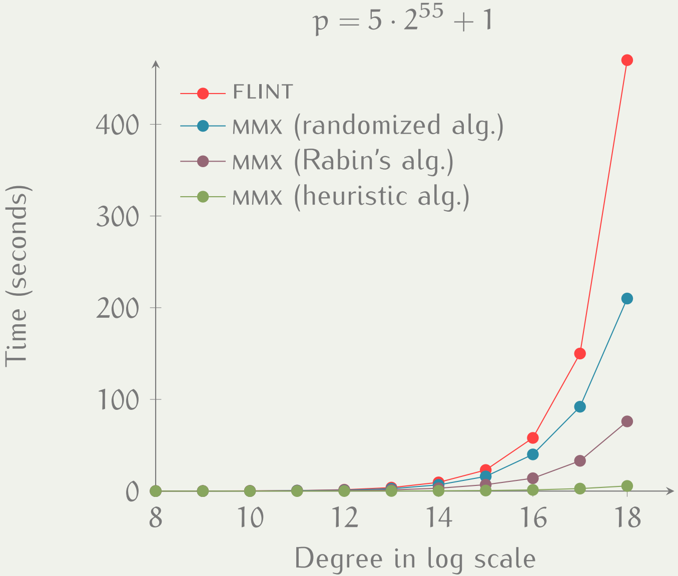
Suppose that f is chosen at random in $\mathbb{F}_p[X]$ or that the heuristic holds. Given a primitive element of $\mathbb{F}_p^*$, the heuristic algorithm runs in **expected time** $\tilde{O}(d \log^2 p)$, for $p = M \cdot 2^m + 1$ with $M = O(\log p)$.

Timings



$p = 7 \cdot 2^{26} + 1$
FLINT
NTL
MMX (randomized alg.)
MMX (heuristic alg.)
Time (seconds)
0
5
10
15
20
25
8
10
12
14
16
18
Degree in log scale

Timings


$p = 5 \cdot 2^{55} + 1$
FLINT
MMX (randomized alg.)
MMX (Rabin's alg.)
MMX (heuristic alg.)
Time (seconds)
400
300
200
100
0
8
10
12
14
16
18
Degree in log scale

Conclusion

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  - Good probabilistic complexity bounds
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Merci de votre attention !