Representations of polynomials, algorithms and lower bounds

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ÉNS Lyon & IRMAR (Rennes)

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Introduction

Polynomials
(formal)

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Lists

- Coefficients $\rightsquigarrow$ dense
- Monomials $\neq 0$ $\rightsquigarrow$ sparse

$\rightsquigarrow$ lacunary

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- Circuits, formulas
- Branching programs
- Graphs (determinants)

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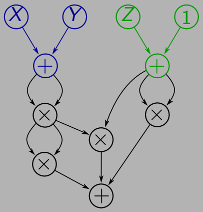

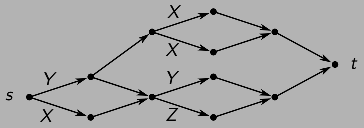

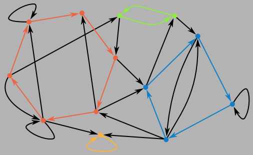

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Algebraic "P = NP ?"
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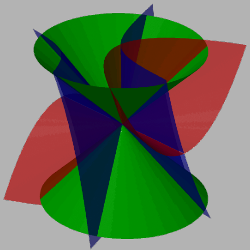
Symbolic Computation | **Algebraic Complexity** | **Applications**

Outline

1. Resolution of polynomial systems
2. Determinantal Representations of Polynomials
3. Factorization of lacunary polynomials

1. Resolution of polynomial systems

Is there a (nonzero) solution?



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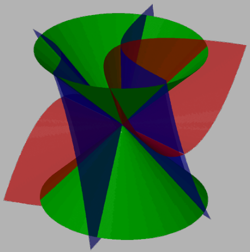

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Question: Is there a point $a = (a_1, a_2, a_3) \in \mathbb{C}^3$, **nonzero**, s.t. $f_1(a) = f_2(a) = f_3(a) = 0$?

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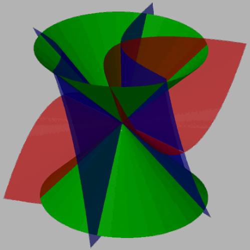

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More on the homogeneous case

Input: $f_1, \dots, f_s \in \mathbb{K}[X_0, \dots, X_n]$, homogeneous

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  $\rightsquigarrow$ Trivial? Easy? Hard?

Definitions

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Proposition (Koiran'96)

Under the Generalized Riemann Hypothesis, $\textsc{PolSys}(\mathbb{Z}) \in \mathsf{AM}$.

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Corollary

Under GRH, HomPolSys($\mathbb{Z}$) and Resultant($\mathbb{Z}$) belong to AM.

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Positive characteristics

If p is prime, $(\textsc{Hom})\textsc{PolSys}(\mathbb{F}_p)$ & $\textsc{Resultant}(\mathbb{F}_p)$ are in PSPACE.

Known lower bounds

Notation: $\mathbb{F}_0 = \mathbb{Q}$

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For $p = 0$ or prime, PolSys($\mathbb{F}_p$) & HomPolSys($\mathbb{F}_p$) are **NP-hard**.

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- What happens for Resultant($\mathbb{F}_p$), $p > 0$?

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Theorem (G.-Koiran-Portier'10-12)

Let p be a prime number.

- Resultant($\mathbb{F}_p$) is NP-hard for **sparse** polynomials.
- Resultant($\mathbb{F}_q$) is NP-hard for **dense** polynomials for some $q = p^s$.

Proof idea

$f(X)$: s degree-2 homogeneous polynomials in $\mathbb{F}_p[X_0, \dots, X_n]$

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From $f(X)$ to $g(X,Y)$

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- $f(a) = 0 \implies g(a, 0) = 0$

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- **NP-hardness results** for square homogeneous systems of polynomials over finite fields

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Main open problem

- Improve the PSPACE upper bound in positive characteristics...
- ... or the NP lower bound.

2. Determinantal Representations of Polynomials

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
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- Complexity of the determinant

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- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "P = NP?"

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0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "P = NP?"
- Links between circuits, ABPs and the determinant

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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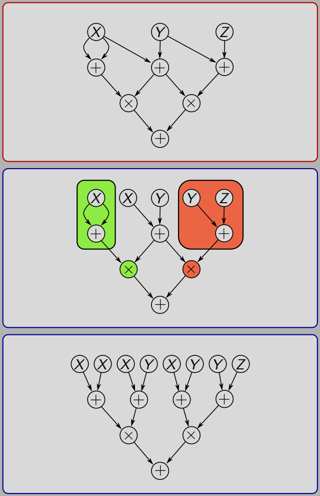
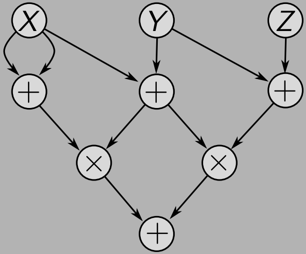
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "$P = NP$?"
- Links between circuits, ABPs and the determinant
- Convex optimization

Circuits



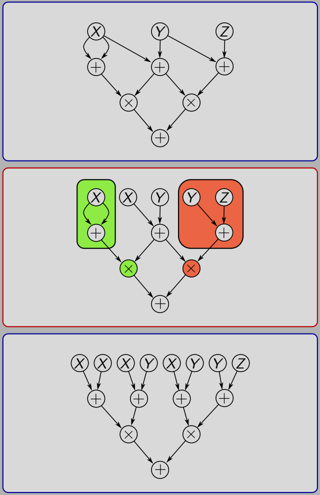
$$2X(X+Y)+(X+Y)(Y+Z)$$



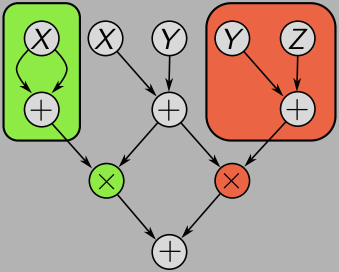
Arithmetic circuit

Size	6
Inputs	3

Circuits



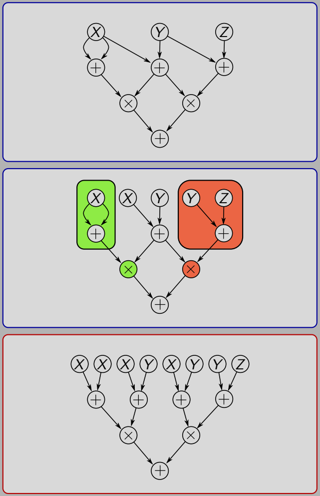
$$2X(X+Y)+(X+Y)(Y+Z)$$



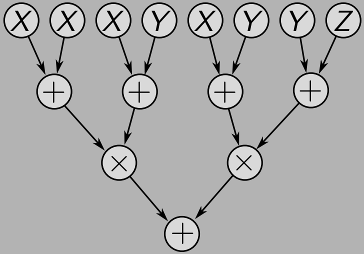
Weakly-skew circuit

Size	6
Inputs	5

Circuits



$$2X(X+Y)+(X+Y)(Y+Z)$$



Formula

Size	7
Inputs	8

Results

Proposition (Valiant'79)

Formula of **size** $s \rightsquigarrow$ Determinant of a matrix of **dimension** $(s+2)$

Results

Proposition (Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11)

Formula of size $s \rightsquigarrow$ Determinant of a matrix of dimension $(s+1)$

Results

Proposition (Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11)

Formula of size s $\rightsquigarrow$ Determinant of a matrix of dimension $(s+1)$

Proposition (Toda'92, Malod-Portier'08)

Weakly-skew circuit of size s with i inputs
$\rightsquigarrow$ Determinant of a matrix of dimension $(s+i+1)$

Results

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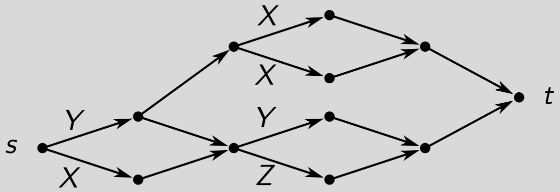

Results

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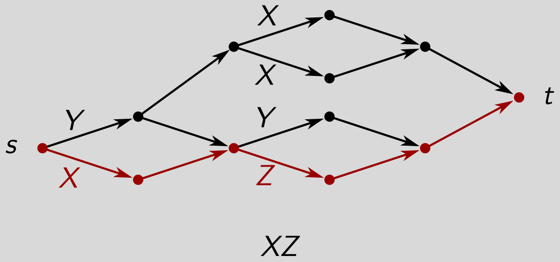
Proposition (Toda'92, Malod-Portier'08)

Weakly-skew circuit of size s with i inputs

$\rightsquigarrow$ Determinant of a matrix of dimension $(s+i+1)$



Results

Proposition (Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11)

Formula of size s $\rightsquigarrow$ Determinant of a matrix of dimension $(s+1)$

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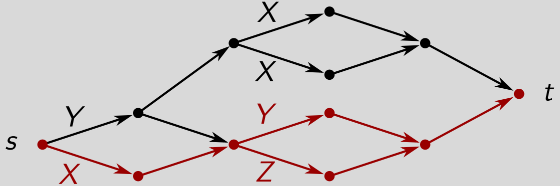


$X(Y+Z)$

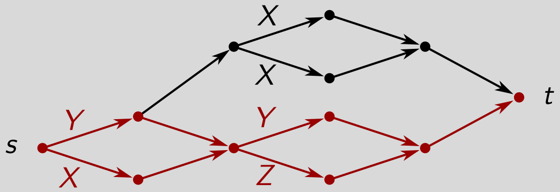
Results

Proposition (Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11)

Formula of size $s \rightsquigarrow$ Determinant of a matrix of dimension $(s+1)$

Proposition (Toda'92, Malod-Portier'08)

Weakly-skew circuit of size s with i inputs
$\rightsquigarrow$ Determinant of a matrix of dimension $(s+i+1)$



$(X+Y)(Y+Z)$

Results

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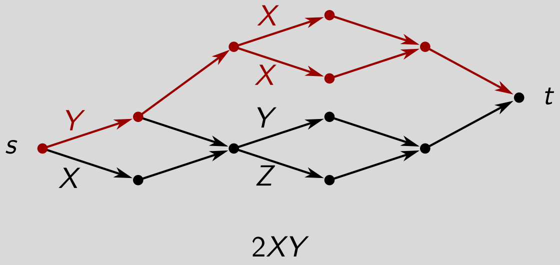

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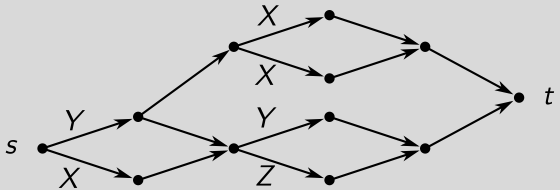
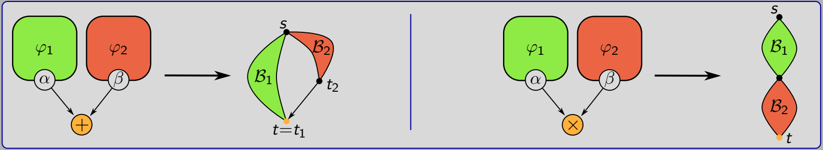


$$2XY + (X+Y)(Y+Z)$$

From Formulas to Branching Programs



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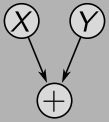
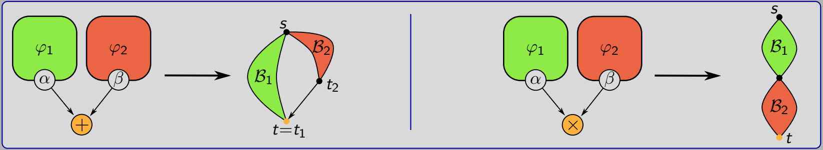
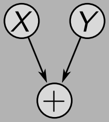
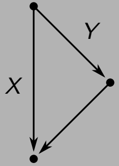
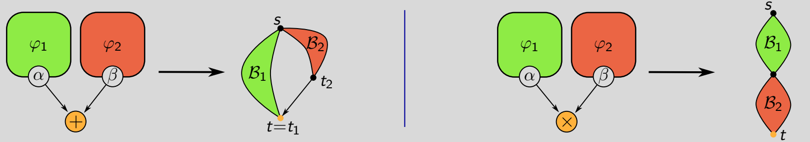




From Formulas to Branching Programs







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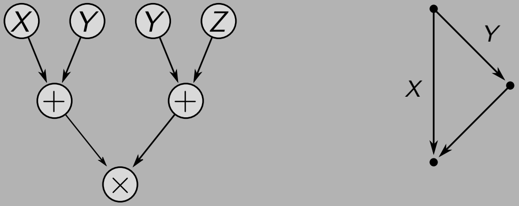

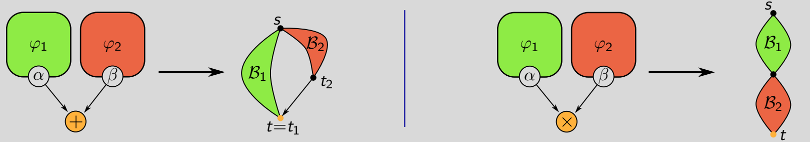

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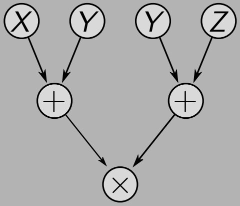
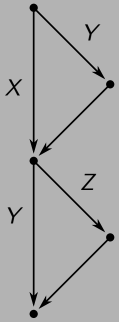
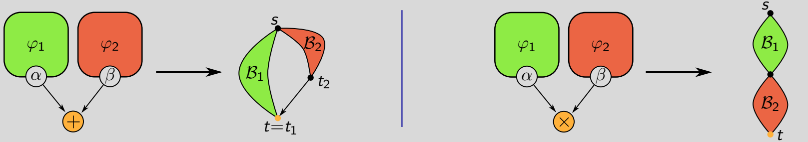
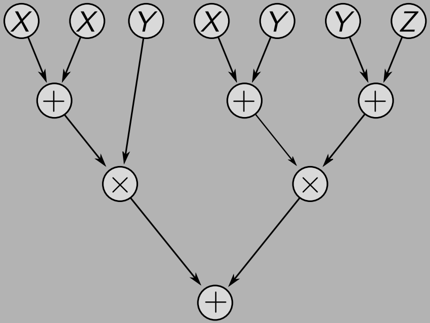
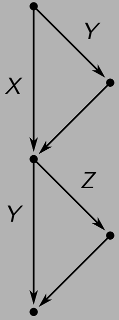


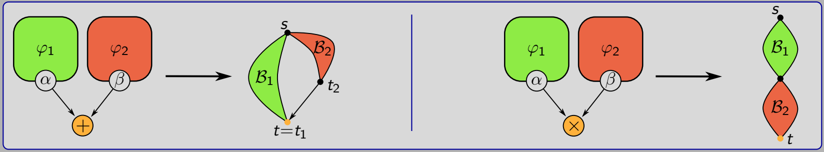




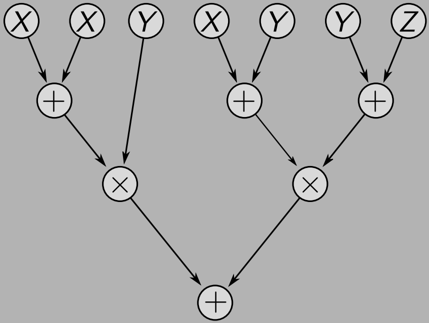
From Formulas to Branching Programs

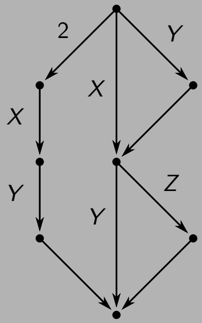


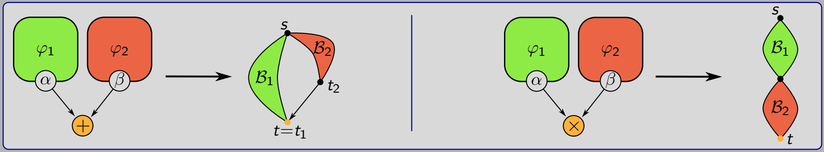




From Formulas to Branching Programs







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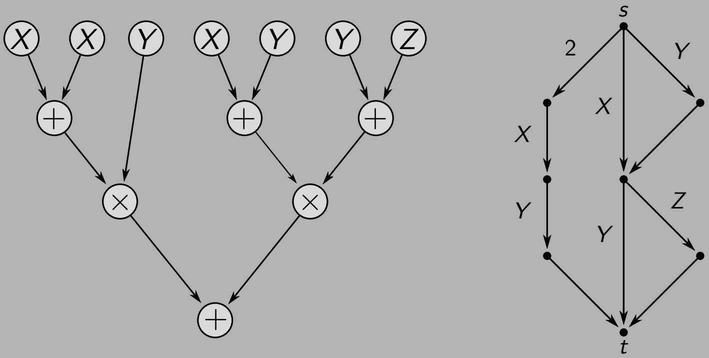
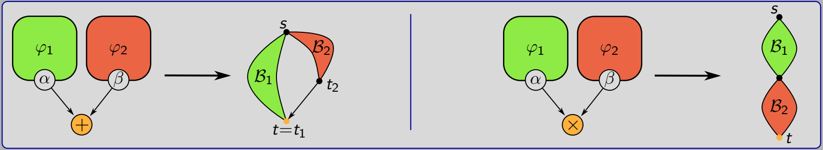
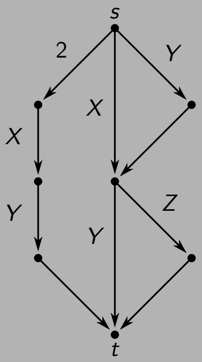




From Branching Programs to Determinants



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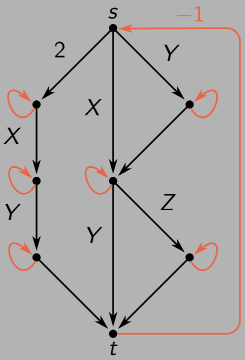

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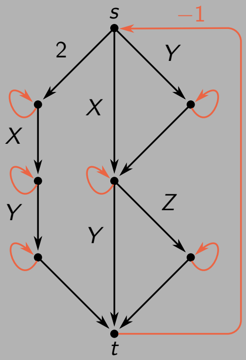
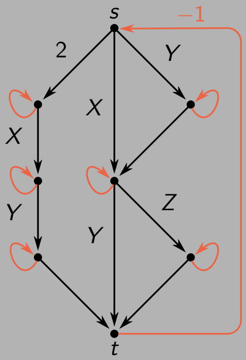


$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From Branching Programs to Determinants



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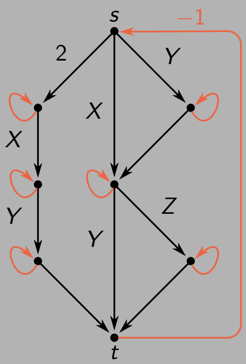
$$\det M = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} M_{i,\sigma(i)}$$

From Branching Programs to Determinants



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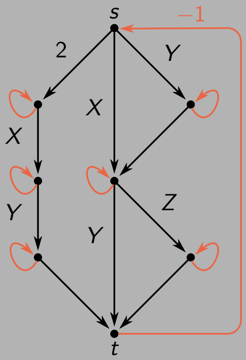
- **Cycle covers** $\iff$ **Permutations**

From Branching Programs to Determinants



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- **Cycle covers** $\iff$ **Permutations**
- Up to signs, $\det(M) =$ **sum of the weights** of the cycle covers of G

Branching Program for the Permanent

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Branching Program for the Permanent

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Theorem (G.'12)

There exists a **branching program of size 2^n** representing the **permanent of dimension n**.

Branching Program for the Permanent

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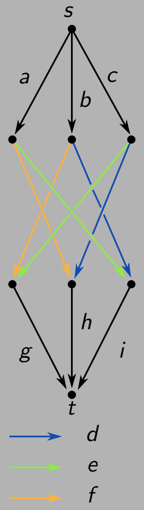

Permanent versus Determinant

Corollary

The **permanent of dimension n** is a projection of the **determinant of dimension $N = 2^n - 1$**.

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$$\mathrm{per}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \det\begin{pmatrix} 0 & a & d & g & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & i & f & 0 \\ 0 & 0 & 1 & 0 & 0 & c & i \\ 0 & 0 & 0 & 1 & c & 0 & f \\ e & 0 & 0 & 0 & 1 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Permanent versus Determinant

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Conjecture (Algebraic P $\neq$ NP)

The **permanent of dimension n** is **not** a projection of the **determinant of dimension $N = n^{\mathcal{O}(1)}$**.

Permanent versus Determinant

Corollary

The **permanent of dimension n** is a projection of the **determinant of dimension $N = 2^n - 1$**.

$$\operatorname{per}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \det\begin{pmatrix} 0 & a & d & g & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & i & f & 0 \\ 0 & 0 & 1 & 0 & 0 & c & i \\ 0 & 0 & 0 & 1 & c & 0 & f \\ e & 0 & 0 & 0 & 1 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Conjecture (Algebraic $\mathrm{P} \neq \mathrm{NP}$)

The **permanent of dimension n** is **not** a projection of the **determinant of dimension $N = 2^{o(n)}$**.

Results

Proposition (Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11)

Formula of size s $\rightsquigarrow$ Determinant of a matrix of dimension $(s+1)$

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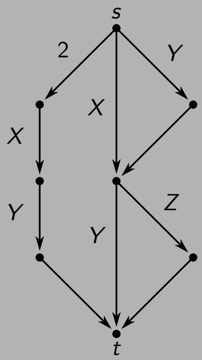
From Branching Programs to Symmetric Determinants



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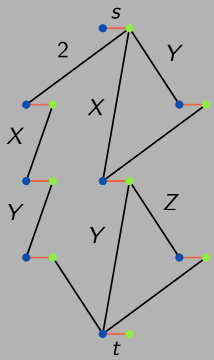

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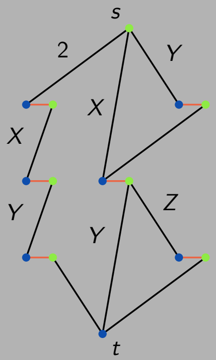

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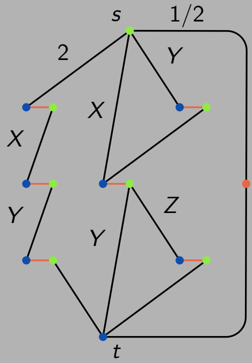

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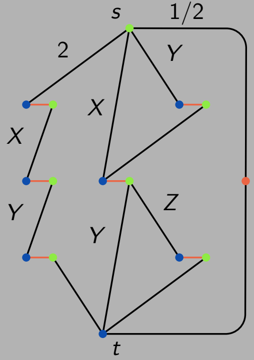

$$S = \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
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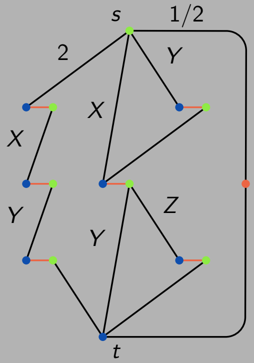

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SDR in characteristic 2

$xy + yz + xz$

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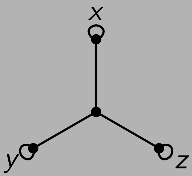

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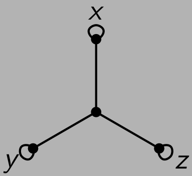

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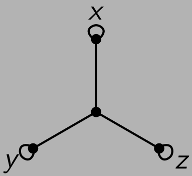

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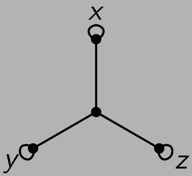

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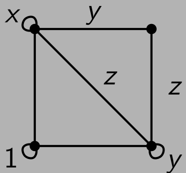

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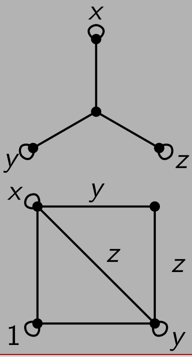

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There are polynomials **without SDR** in characteristic 2, e.g. $xy+z$.

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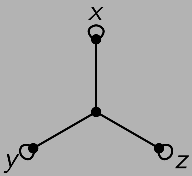

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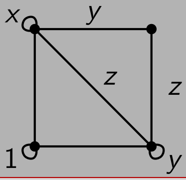

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A polynomial is said **representable** if it has an SDR.

Determinant and cycle covers

Determinant

$\mathfrak{S}_n$ = Permutation group of $\{1, \ldots, n\}$

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

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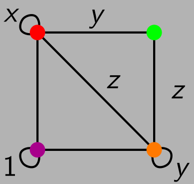

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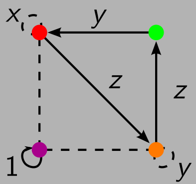

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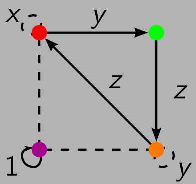

Determinant and partial matchings

Determinant in characteristic 2 of symmetric matrices

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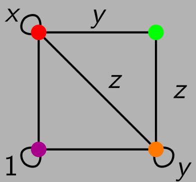

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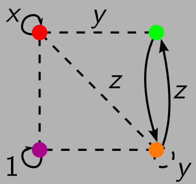

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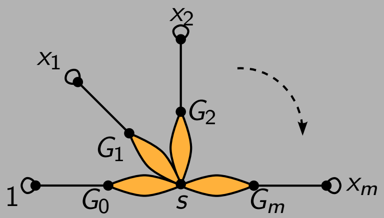

Obstructions to representability

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If P is representable, then

$$P \equiv L_1 \times \cdots \times L_k \bmod \langle x_1^2 + 1, \ldots, x_m^2 + 1 \rangle$$

where the L_i's are linear. (linear = degree-1)

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Such a P is said **factorizable** *modulo* $\langle x_1^2 + \ell_1^2, \ldots, x_m^2 + \ell_m^2 \rangle$.

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$\rightsquigarrow$ Factorization algorithm for $\mathbb{F}[x_1, \dots, x_m]/\langle x_1^2 + \ell_1^2, \dots, x_m^2 + \ell_m^2 \rangle$

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Theorem

Under *suitable* conditions, P is factorizable if and only if

$$P \equiv \mathrm{lin}(P) \times \frac{1}{\alpha_i}\frac{\partial P}{\partial x_i} \mod \langle x_1^2, \dots, x_m^2 \rangle,$$

where $\alpha_i x_i$ is a monomial of $\mathrm{lin}(P)$.

Conclusion

Same **expressiveness**:

- (Weakly-)Skew circuits

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Main open question (Algebraic "P = NP?")

What is the **smallest** N s.t. the **permanent of dimension** n is a projection of the **determinant of dimension** N?

3. Factorization of lacunary polynomials

Introduction

Definition

$$P(X_1,\ldots,X_n)=\sum_{j=1}^{k} a_j X_1^{\alpha_{1j}}\cdots X_n^{\alpha_{nj}}$$

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- Lacunary representation:

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- Size:

$$\mathrm{size}(P)\simeq\sum_{j=1}^{k}\mathrm{size}(a_j)+\log(\alpha_{1j})+\cdots+\log(\alpha_{nj})$$

Factorization: dense/sparse vs. lacunary

Factorization of a polynomial P

Find $F_1, \dots, F_t$, irreducible, s.t. $P = F_1 \times \cdots \times F_t$

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$\Longrightarrow$ restriction to finding **some** factors

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$$P(X) = \sum_{j=1}^{k} a_j X^{\alpha_j} \qquad \mathrm{size}(P) \simeq \sum_{j=1}^{k} \mathrm{size}(a_j) + \log(\alpha_j)$$

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Polynomial-time algorithm to find **integer roots** if $a_j \in \mathbb{Z}$.

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Theorem (Kaltofen-Koiran'06)

Polynomial-time algorithm to find **low-degree factors** of **multivariate** lacunary polynomials over $\mathbb{Q}(\alpha)$.

Common ideas

Gap Theorem

$$P = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j} Y^{\beta_j}}_{P_0} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j} Y^{\beta_j}}_{P_1}$$

with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$.

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gap(P): function of the **algebraic height** of P.

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Polynomial time algorithm to find **multilinear** factors of **bivariate** lacunary polynomials over algebraic number fields.

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Bound on the valuation

Definition

$\mathrm{val}(P)$ = degree of the **lowest degree monomial** of $P \in \mathbb{K}[X]$

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- $X^{\alpha_j}(uX+v)^{\beta_j}$ linearly independent
- Hajós' Lemma: if $\alpha_1 = \cdots = \alpha_k$, $\mathrm{val}(P) \leq \alpha_1 + (k-1)$

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with $uv \neq 0$, $\alpha_1 \leq \cdots \leq \alpha_k$. If

$$\alpha_{\ell+1} > \max_{1\leq j\leq \ell}\left(\alpha_j + \binom{\ell+1-j}{2}\right),$$

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The Wronskian

Definition

Let $f_1, \dots, f_k \in \mathbb{K}[X]$. Then

$$\operatorname{wr}(f_1, \dots, f_k) = \det \begin{bmatrix} f_1 & f_2 & \dots & f_k \\ f_1' & f_2' & \dots & f_k' \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \dots & f_k^{(k-1)} \end{bmatrix}.$$

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Proposition (Bôcher, 1900)

$\mathrm{wr}(f_1, \dots, f_k) \neq 0 \iff$ the f_j's are linearly independent.

Wronskian & valuation

Lemma

$$\mathrm{val}(\mathrm{wr}(f_1,\ldots,f_k)) \geq \sum_{j=1}^{k} \mathrm{val}(f_j) - \binom{k}{2}$$

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$$\sum_{j=1}^{k} \alpha_j \geq \mathrm{val}(\mathrm{wr}(f_1,\dots,f_k)) \geq \mathrm{val}(P) + \sum_{j=2}^{k} \alpha_j - \binom{k}{2}$$

Finding linear factors

Observation + Gap Theorem

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  $\rightsquigarrow$ [Kaltofen'82, ..., Lecerf'07]

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Main open problem

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Thank you!