Generic reductions for in-place polynomial multiplication

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Computer Algebra 101

Multiplication of polynomials: $\mathsf{M}(n)$

Naive:	$O(n^2)$	
Karatsuba:	$O(n^{\log_2 3}) = O(n^{1.585})$	Karatsuba (1962)
Toom-3:	$O(n^{\log_3 5}) = O(n^{1.465})$	Toom (1963), Cook (1966)
FFT-based algorithms:		
	$O(n \log n)$ with $\omega^{2n} = 1$	Cooley, Tukey (1965)
	$O(n \log n \log\log n)$	Cantor, Kaltofen (1991)
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What about space complexity?

Space-complexity models

Algebraic-RAM Machine

- *Standard* registers of size $O(\log n)$
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FFT/TFT-based algorithms (given $\omega^{2n} = 1$)**:**		
Original (1965)	$O(n)$	$\sim 9n \log(2n)$
Roche (2009) if $n = 2^k$	$O(1)$	$\sim 11n \log(2n)$
Harvey, Roche (2010)	$O(1)$	$O(n \log(n))$

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- What about other products (short and middle)?

Results:

- Yes!
- Almost (for other products)

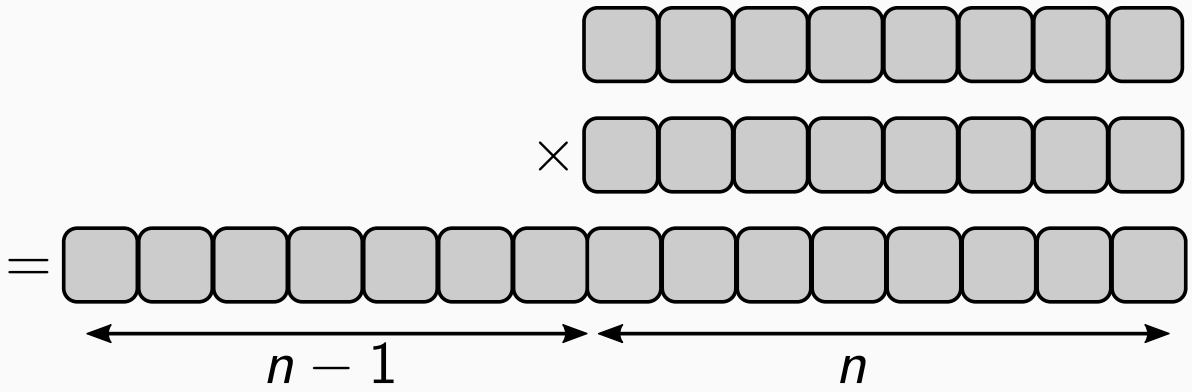
Contents

Space-preserving reductions between products

Self-reductions: In-place algorithms from out-of-place algorithms

Space-preserving reductions between products

Short product



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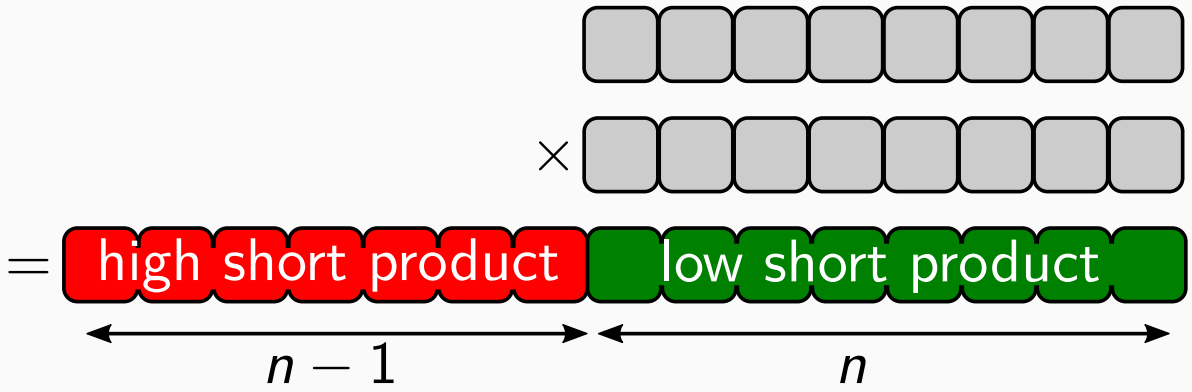

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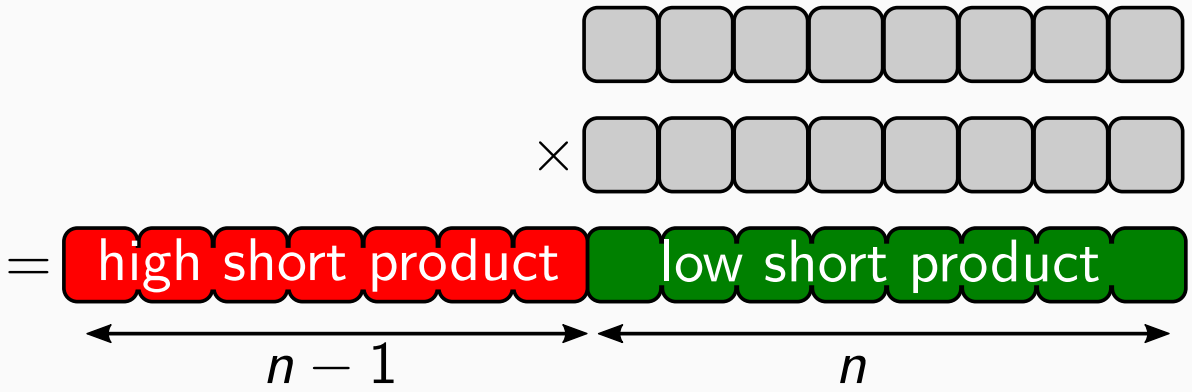
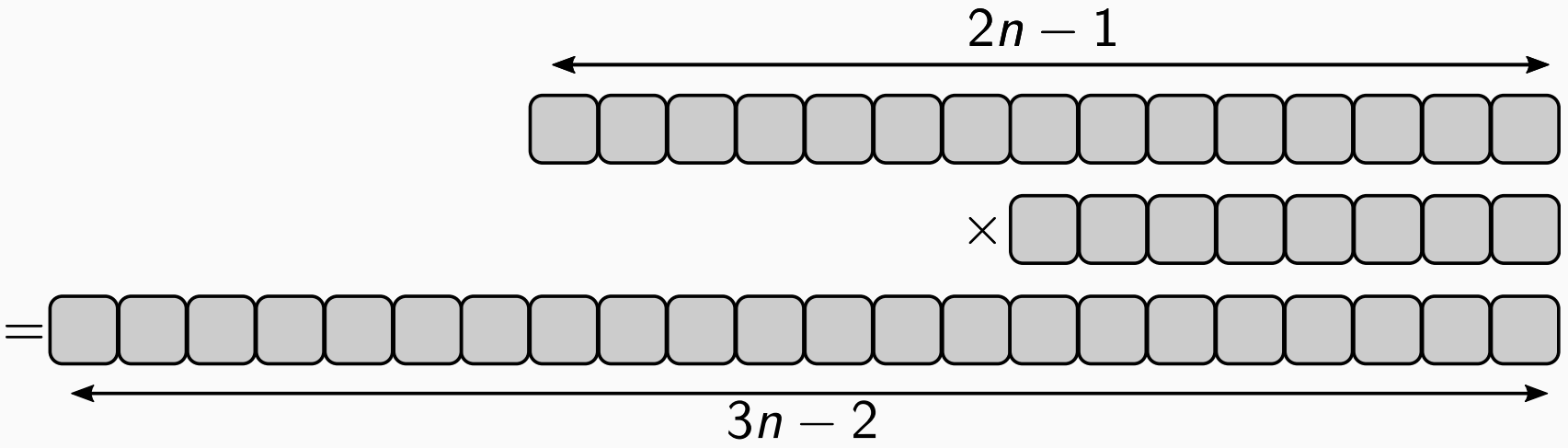


Formal definition

- $\mathrm{SP}_{\mathrm{lo}}(f, g) = f \cdot g \mod X^n$
- $\mathrm{SP}_{\mathrm{hi}}(f, g) = f \cdot g \operatorname{div} X^n$

Example of use
Product of truncated power series

Middle product



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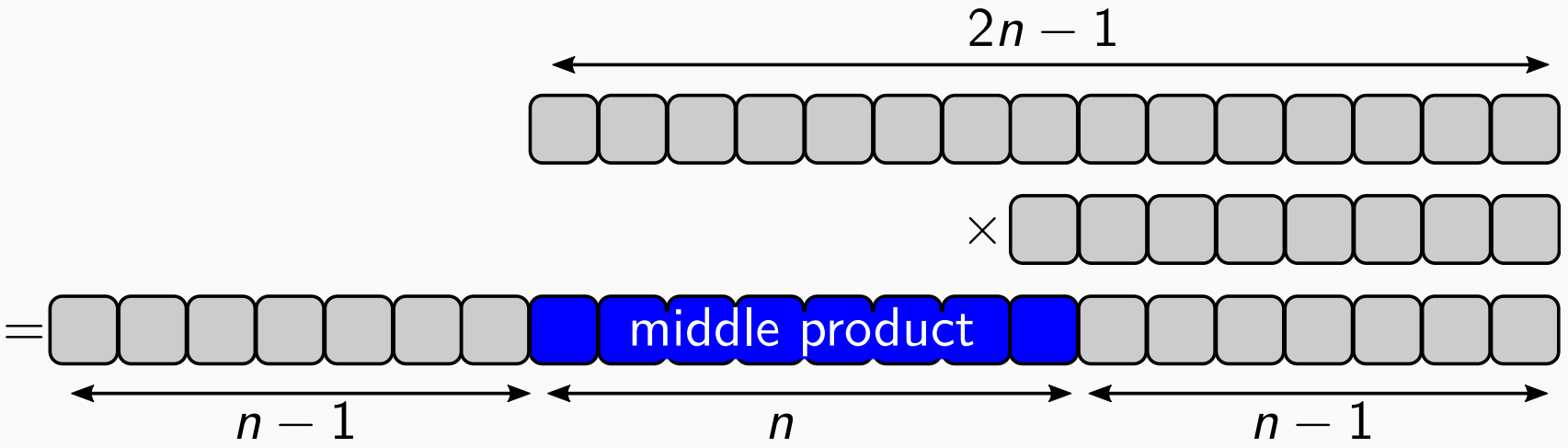

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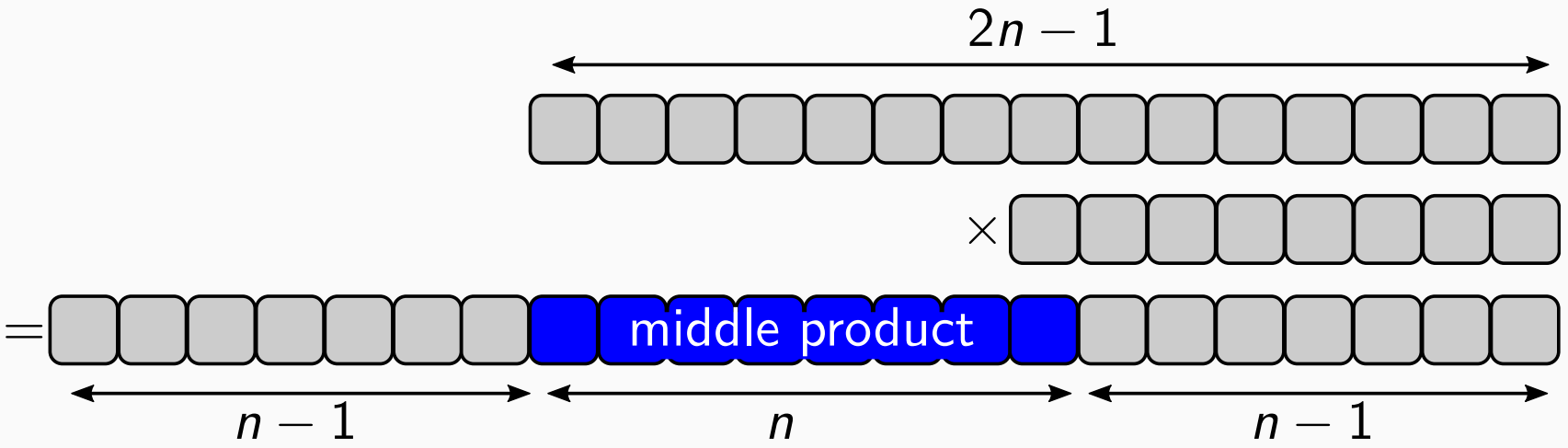

Formal definition

$\mathrm{MP}(f, g) = (f \cdot g \text{ div } X^{n-1}) \mod X^n$

Example of use

Newton iteration (division, square root, ...)

Relative difficulties of products

Notations

- FP: *full* (standard) product
- $\mathrm{FP}(n)$, $\mathrm{SP}_{\mathsf{lo}}(n)$, $\mathrm{SP}_{\mathsf{hi}}(n)$, $\mathrm{MP}(n)$: complexity for size-n inputs

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Classical results (no space restriction)

- $\mathsf{SP}_{\mathsf{lo}}(n) \leq \mathsf{FP}(n)$ and $\mathsf{SP}_{\mathsf{hi}}(n) \leq \mathsf{FP}(n-1)$
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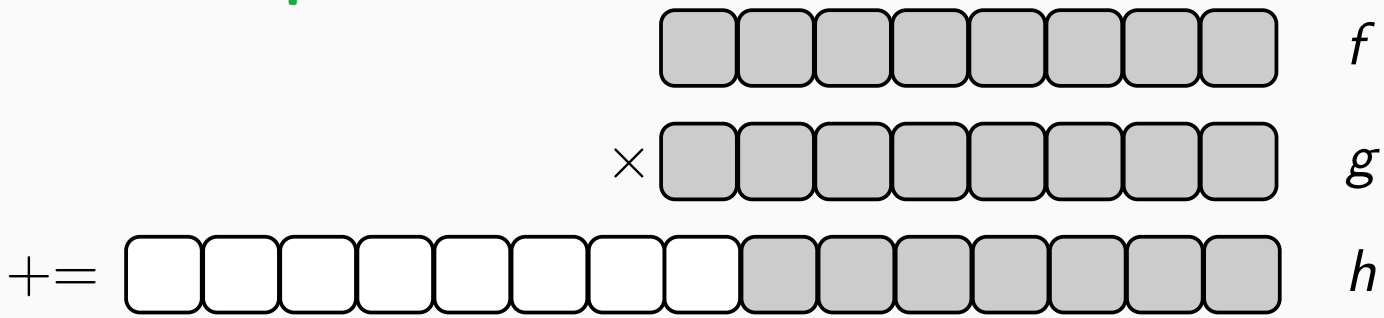
Space-restricted settings

✗ Reductions unusable

✓ Reduction still valid

Relative difficulties in space-restricted settings

Half-additive full product FP^+



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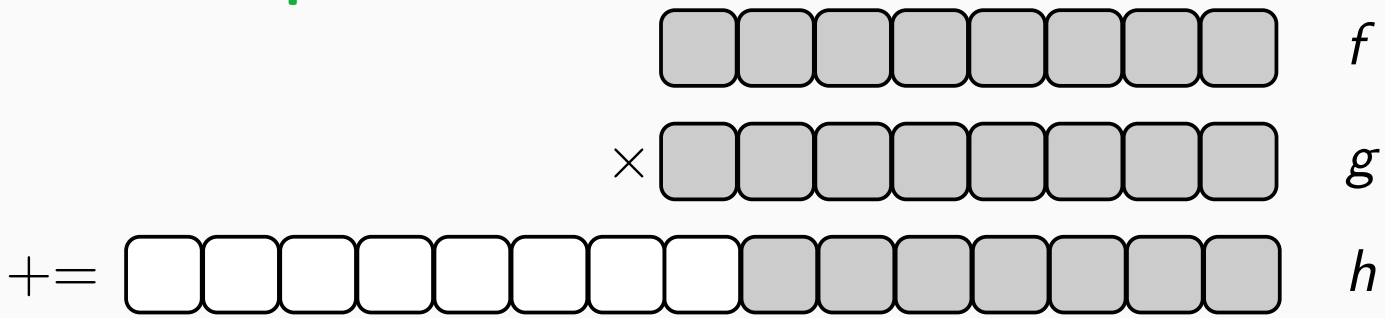

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Theorem ("$\mathsf{SP} \equiv \mathsf{FP}^+$")
The following reductions do not increase space:

- $\mathsf{FP}^+(n) \leq \mathsf{SP}_{\mathsf{lo}}(n) + \mathsf{SP}_{\mathsf{hi}}(n) + n - 1$
- $\mathsf{SP}_{\mathsf{lo}}(n) \leq 3\mathsf{FP}^+(\lceil n/2 \rceil)$

Summary of relative difficulties

Without space restrictions

$$\mathsf{SP} \equiv \mathsf{FP}^{(+)} \equiv \mathsf{MP}$$

In space-restricted settings

$$\mathsf{FP} \leq \begin{pmatrix} \mathsf{SP} \\ \| \\ \mathsf{FP}^{+} \end{pmatrix} \leq \mathsf{MP}$$

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Converse directions

- $\mathsf{SP}(n) \leq \mathsf{FP}(n)$?
  - problem with output size : n v. $2n-1$
  - without space restrictions: is $\mathsf{SP}(n) \leq \mathsf{FP}(n/2)$?
- $\mathsf{FP} \equiv \mathsf{MP}$?
  - partial result: $\mathsf{MP}(n) \leq \mathsf{FP}(n) \times \log(n)$
  - related: space-preserving transposition? Kaltofen (2000)

Self-reductions: In-place algorithms from out-of-place algorithms

Framework

In-place algorithms parametrized by out-of-place algorithms

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- *Fake* padding of inputs (*cf.* strides in lin. alg.)
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Similar approach for matrix mul.: Boyer, Dumas, Pernet, Zhou (2009)

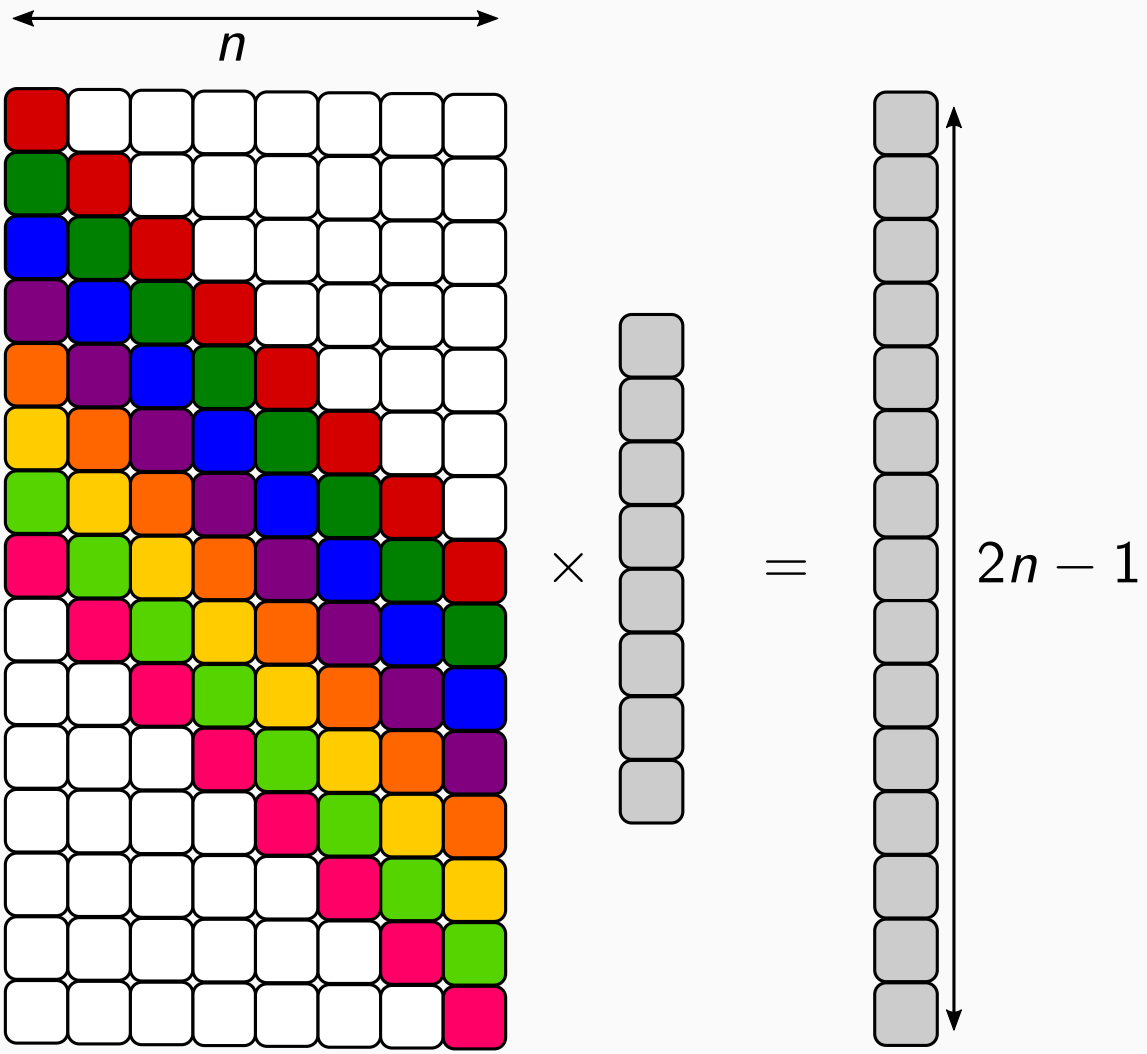
Our results

Theorem

- In-place (half-additive) full product in time $(2c + 7)\mathrm{M}(n)$
- In-place short product in time $(2c + 5)\mathrm{M}(n)$
- In-place middle product in time $O(\mathrm{M}(n) \log n)$
  (or $O(\mathrm{M}(n))$ if $\mathrm{M}(n) = \Omega(n^{1+\delta})$)

Multiplications as linear maps

Full product:



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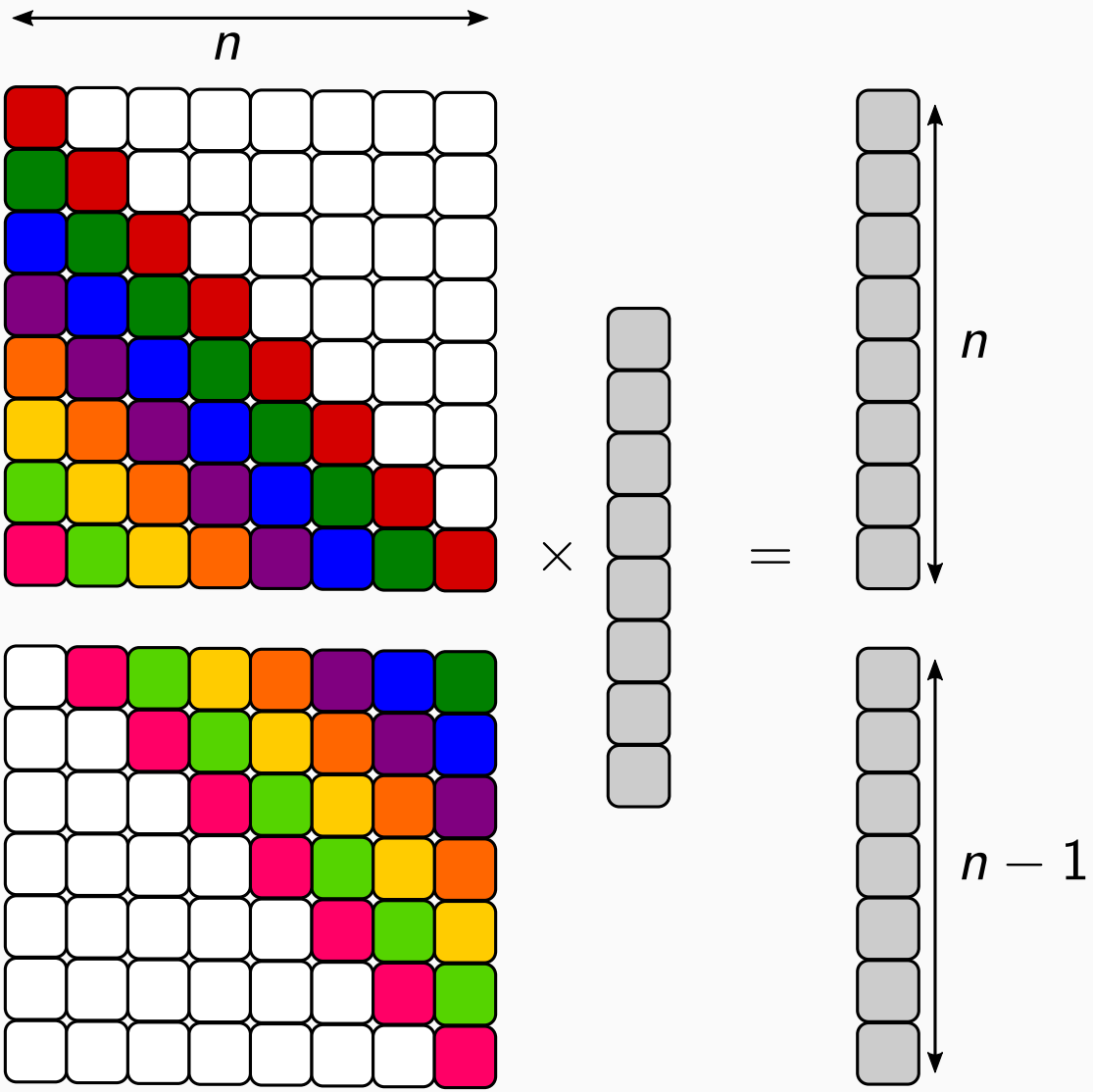

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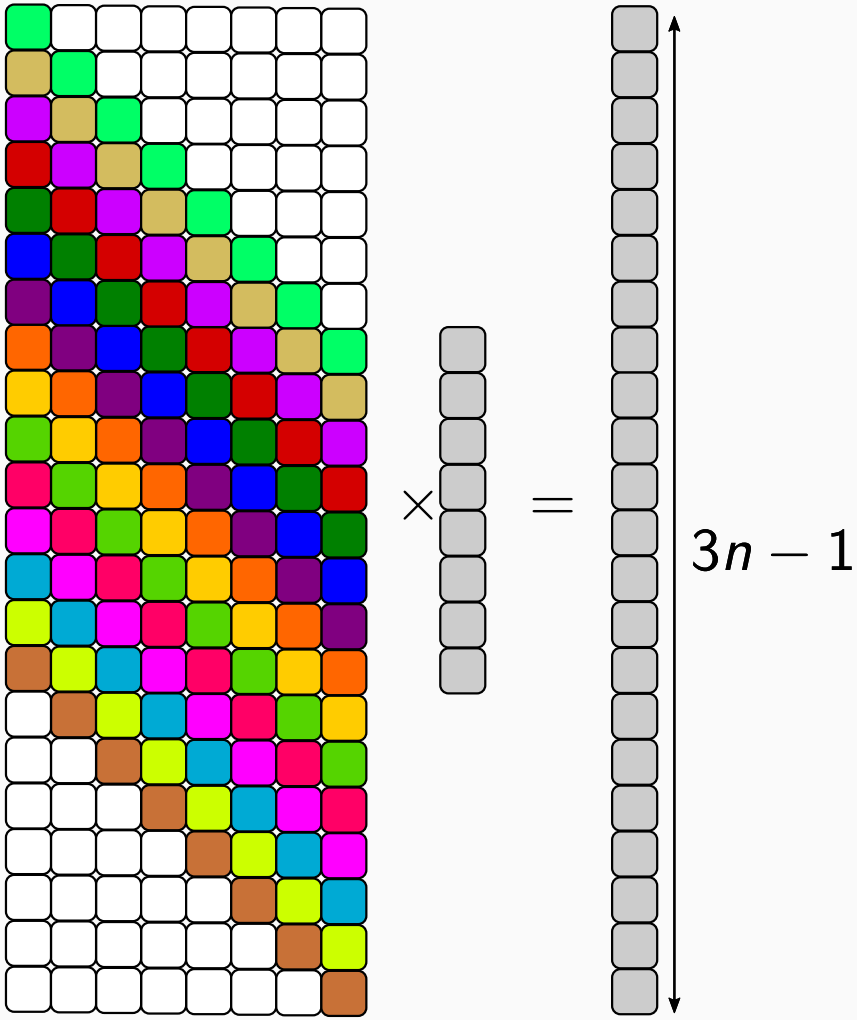

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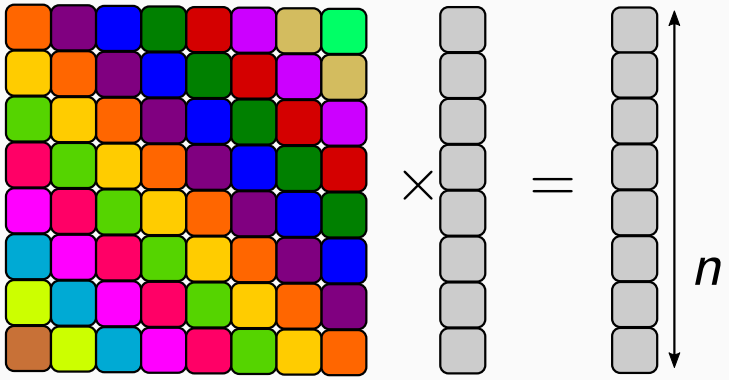

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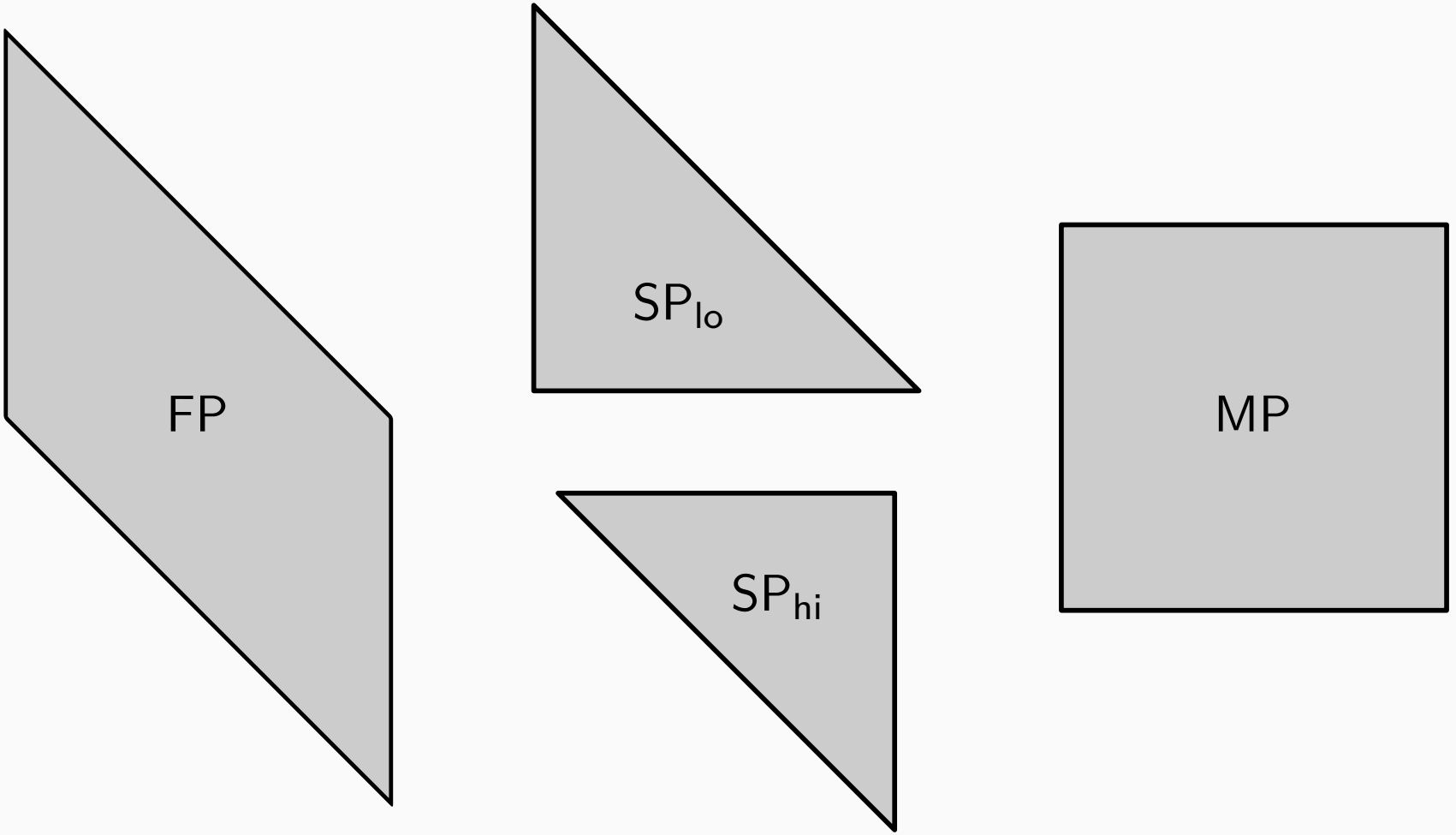
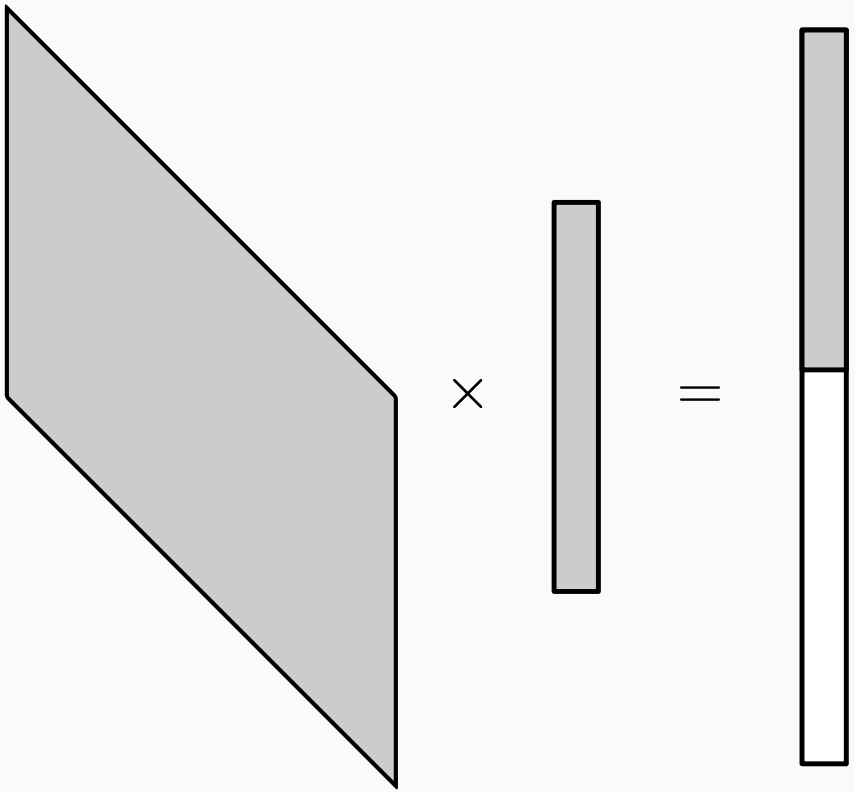
Graphical representation:



Full product Short products Middle product

In-place FP^+ from out-of-place FP



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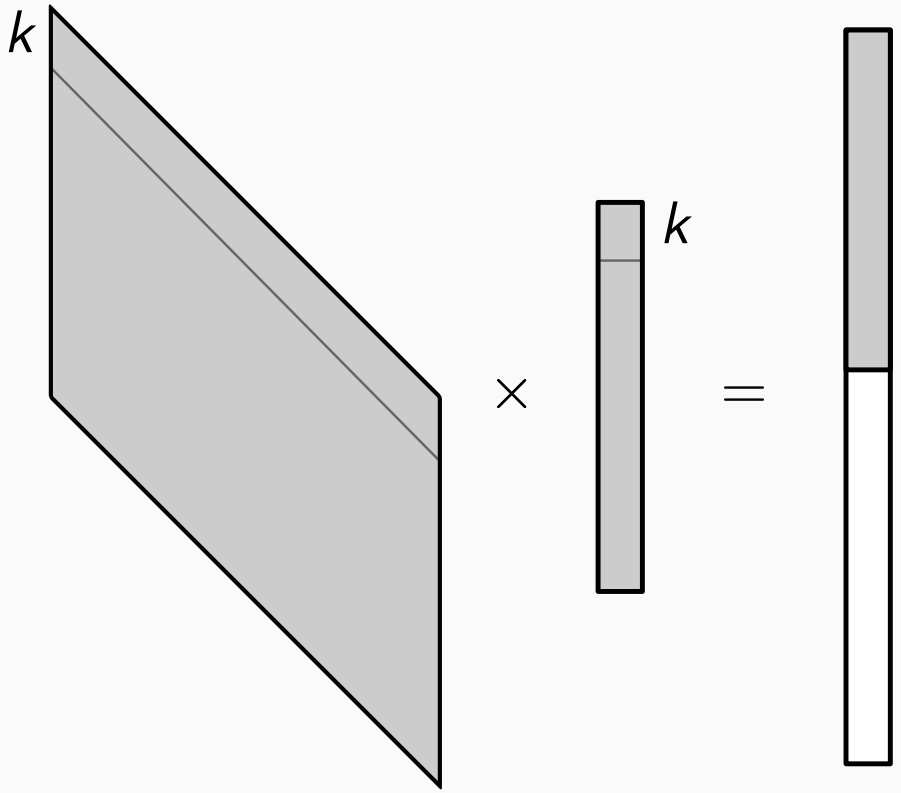

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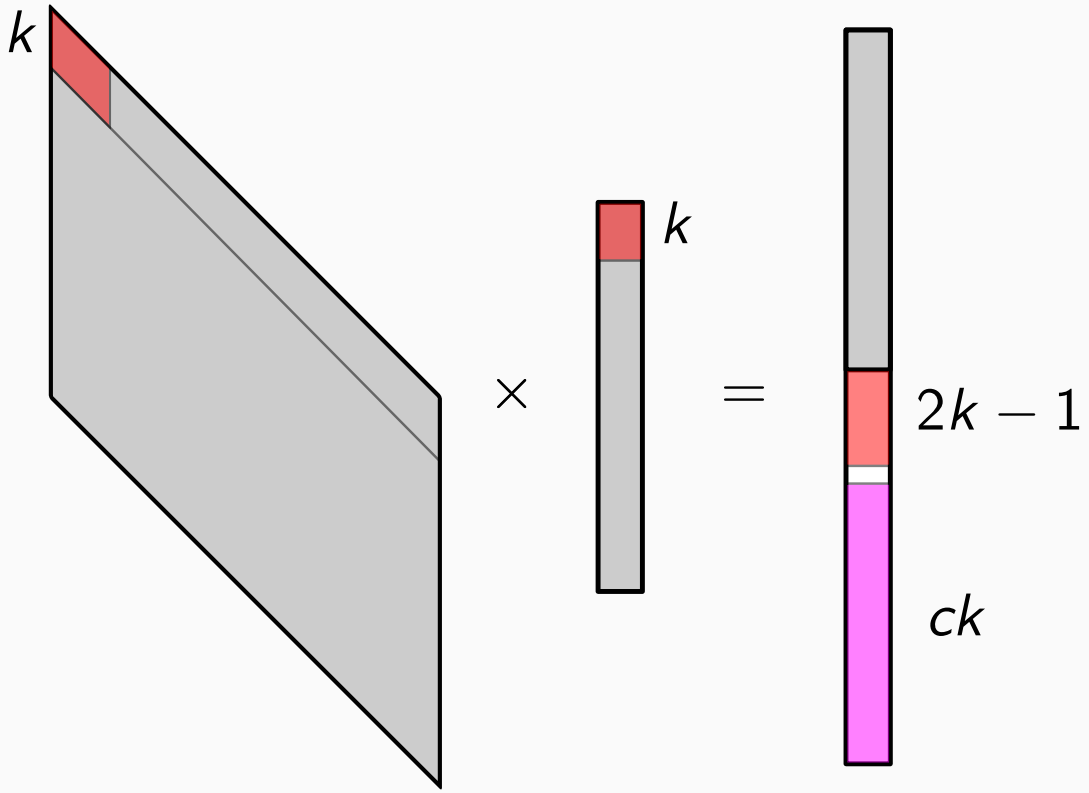

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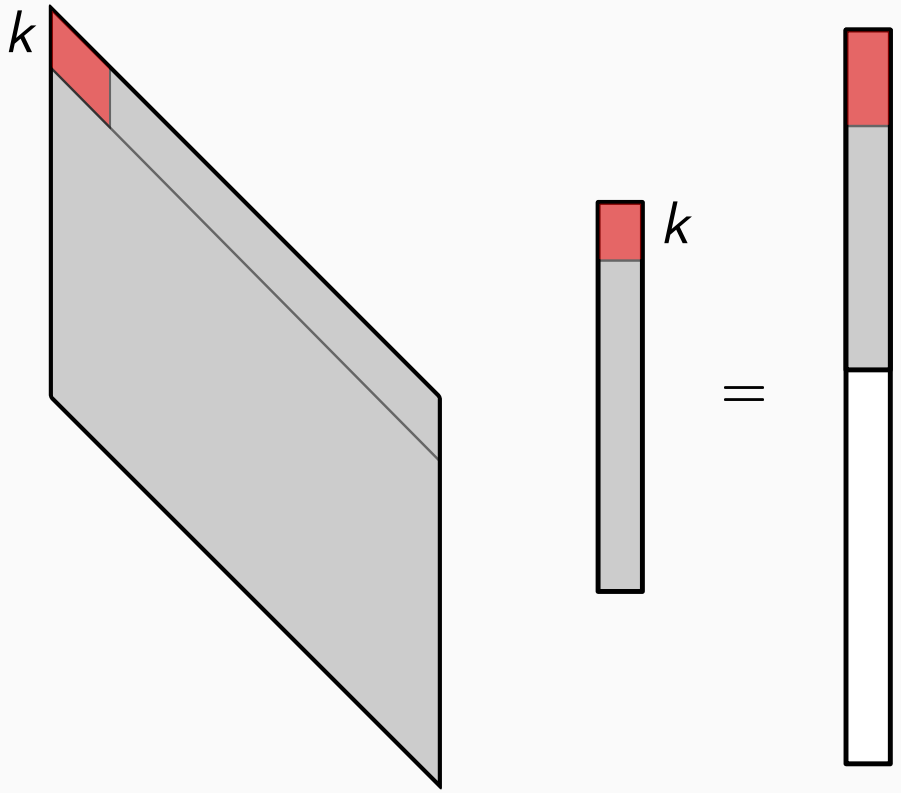

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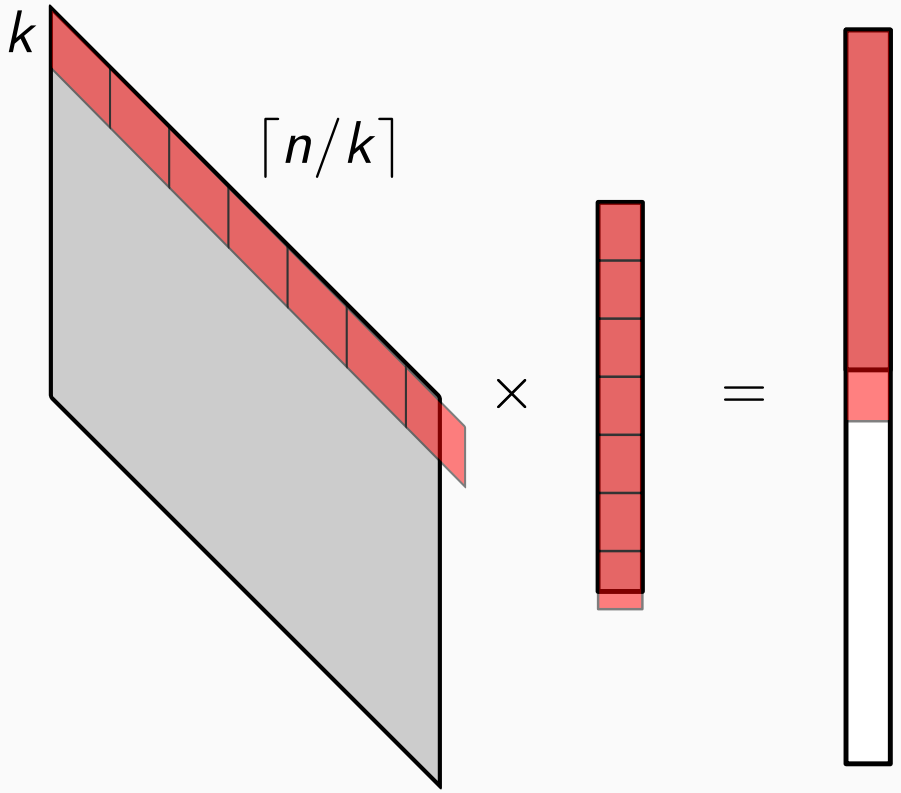

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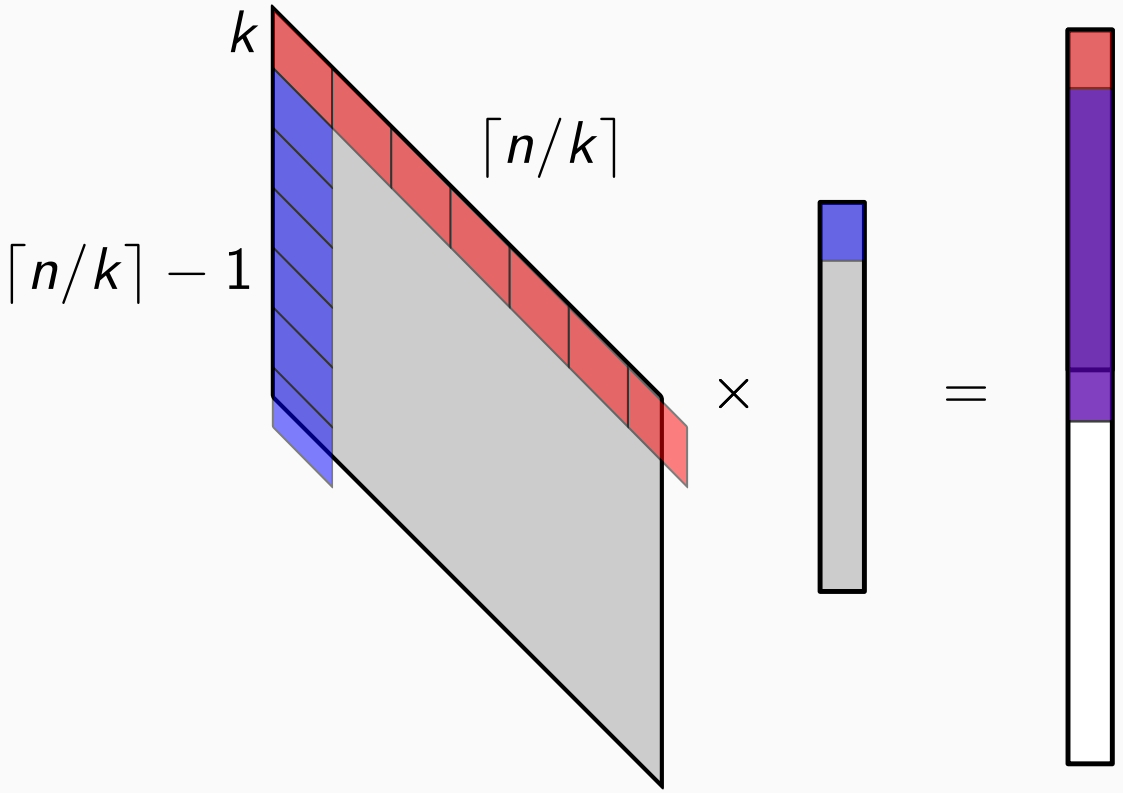

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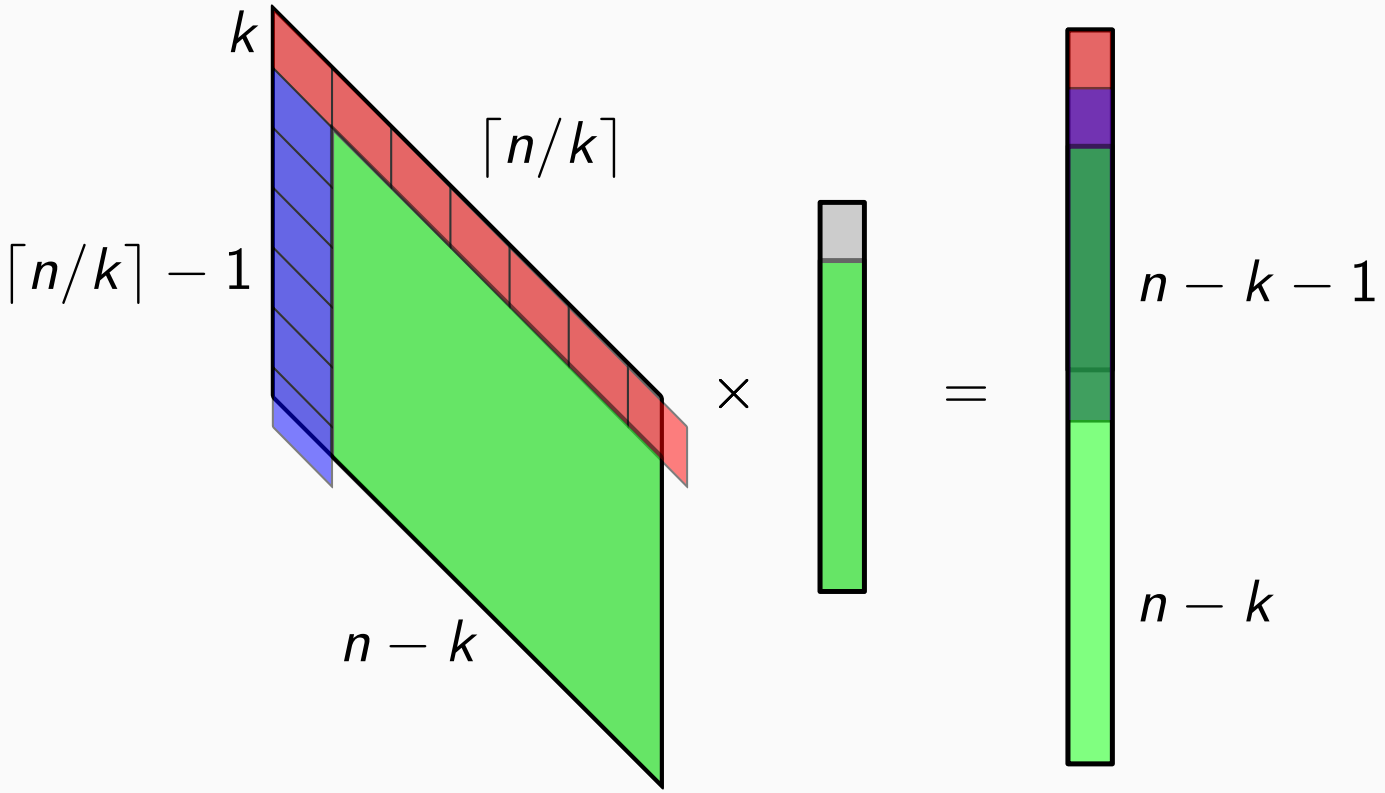

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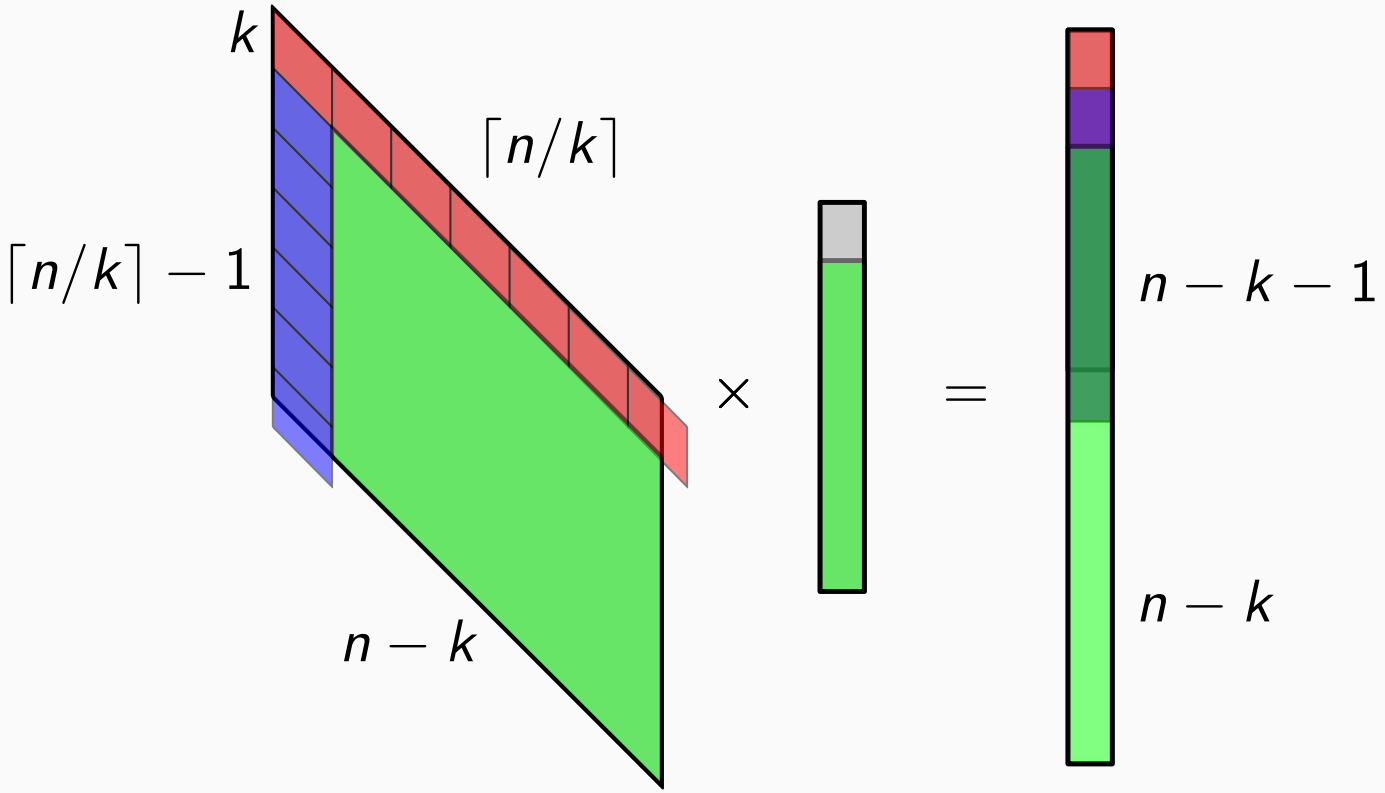

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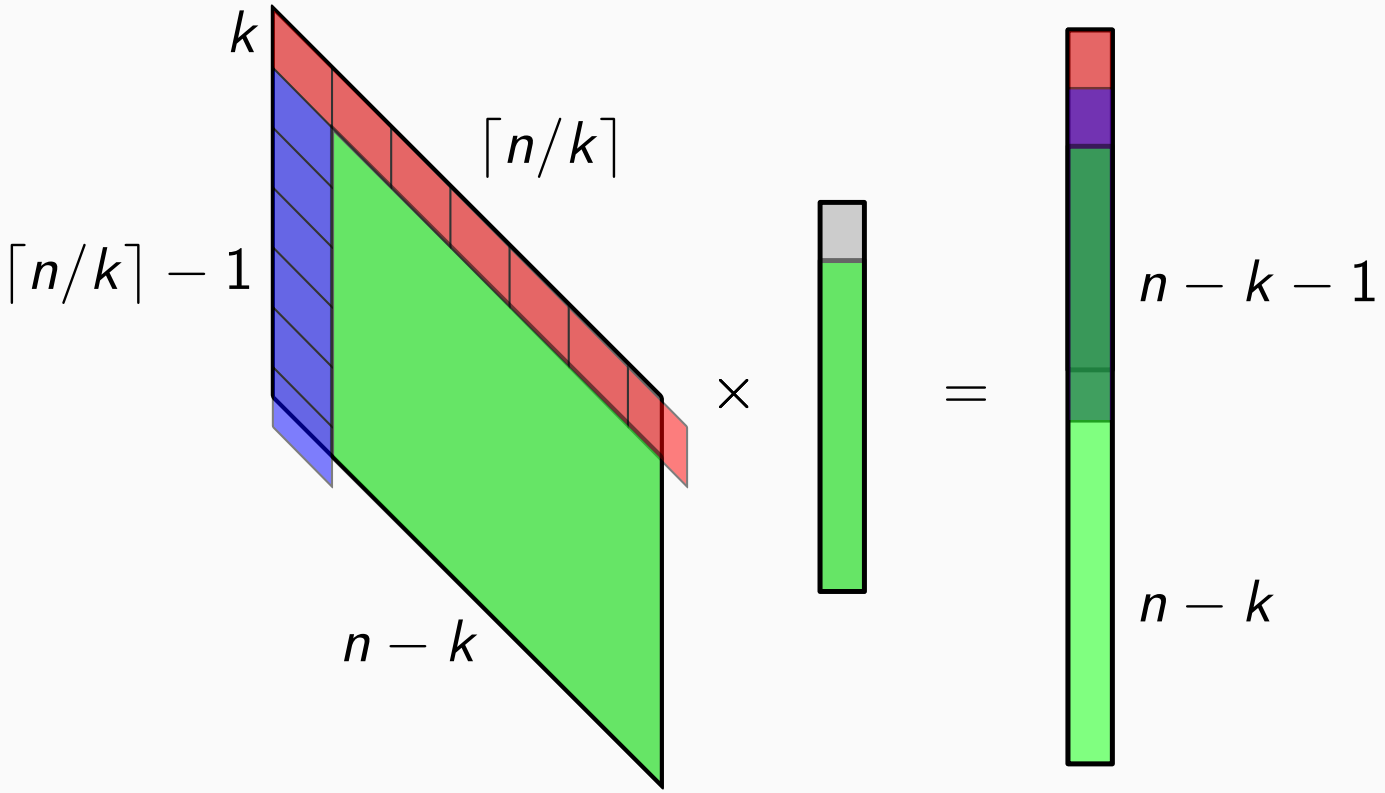

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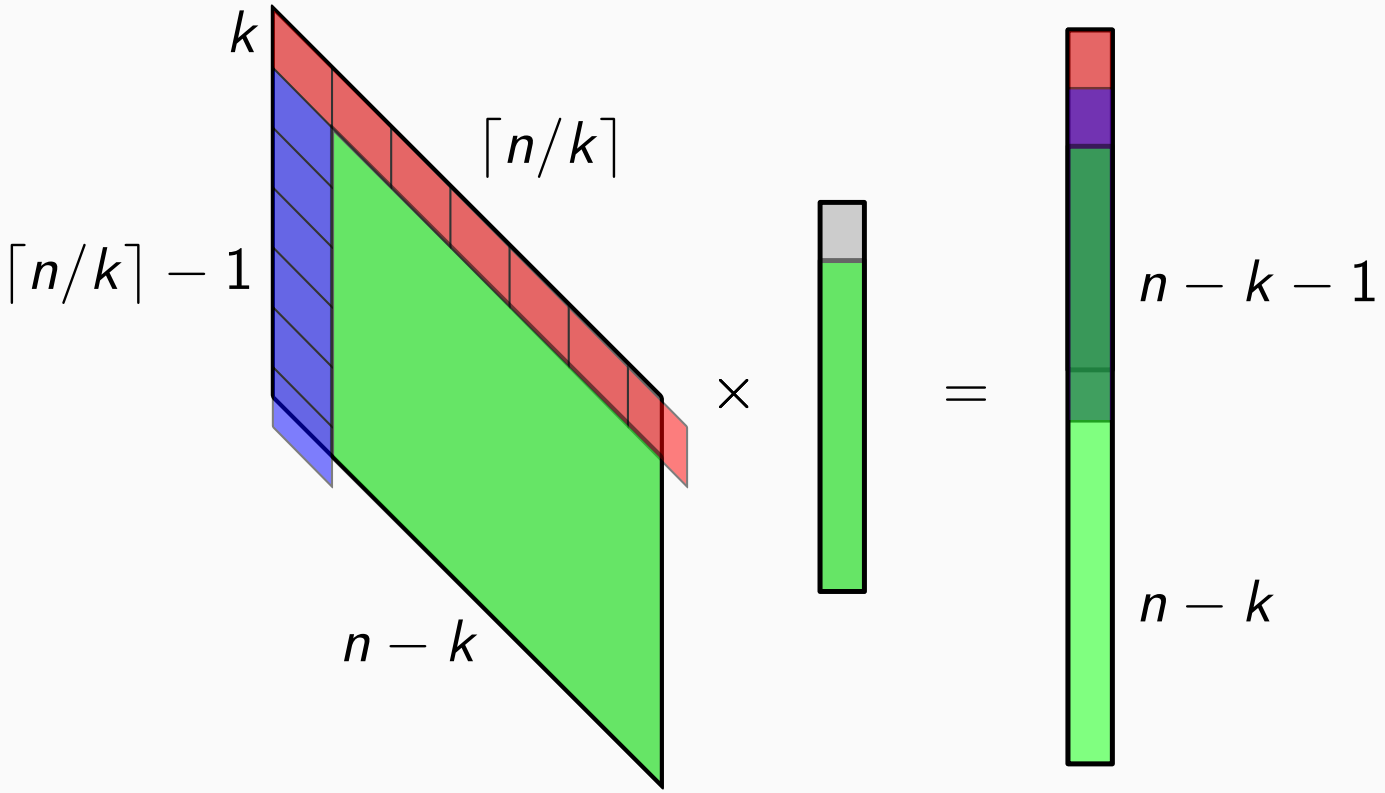

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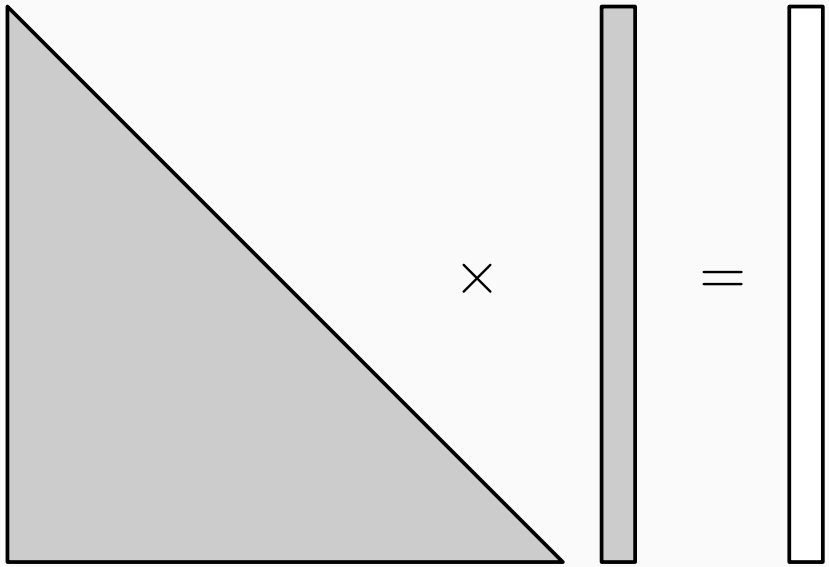
$$T(n) \leq (2c + 7)\mathrm{M}(n) + o(\mathrm{M}(n))$$

In-place short product



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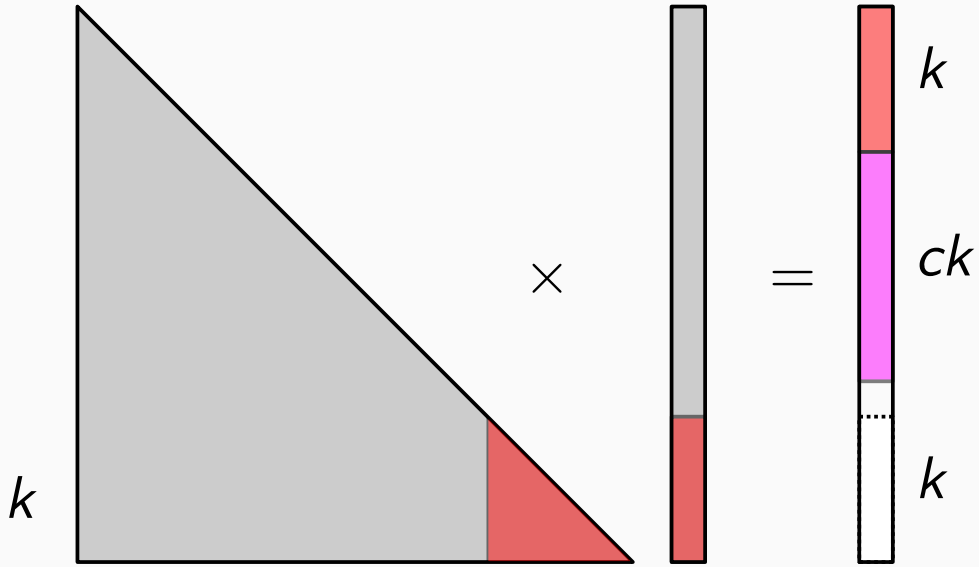

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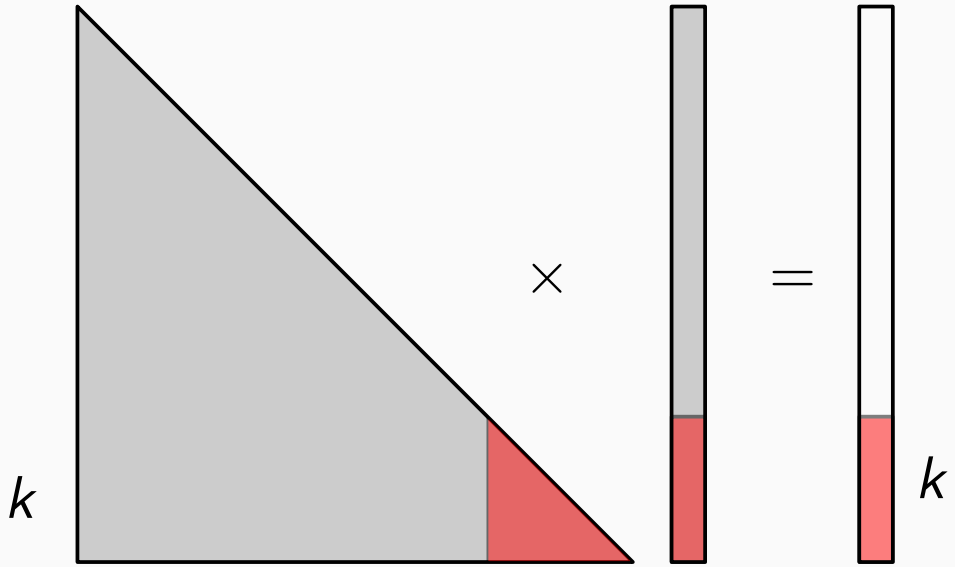

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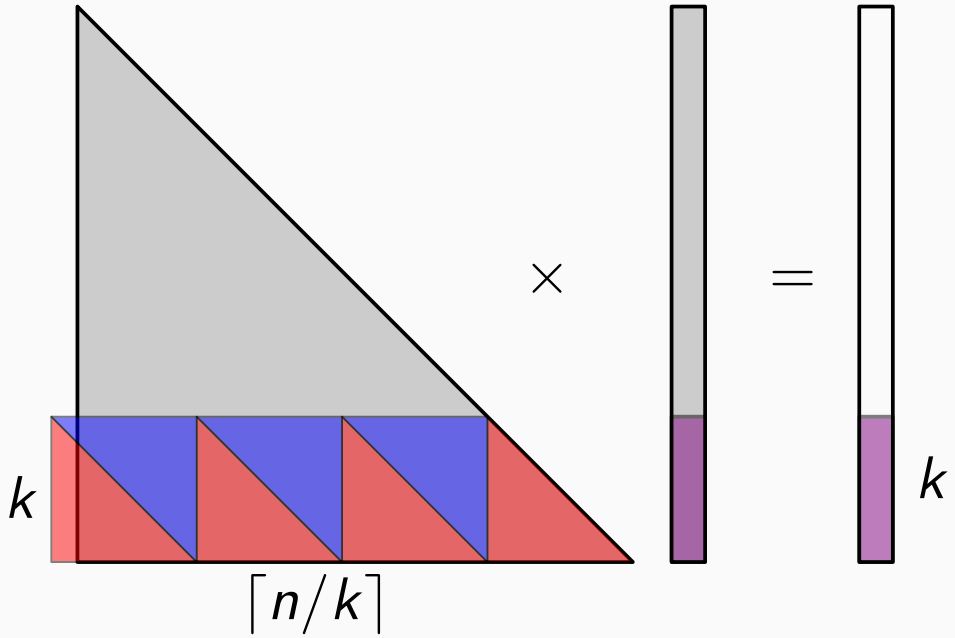

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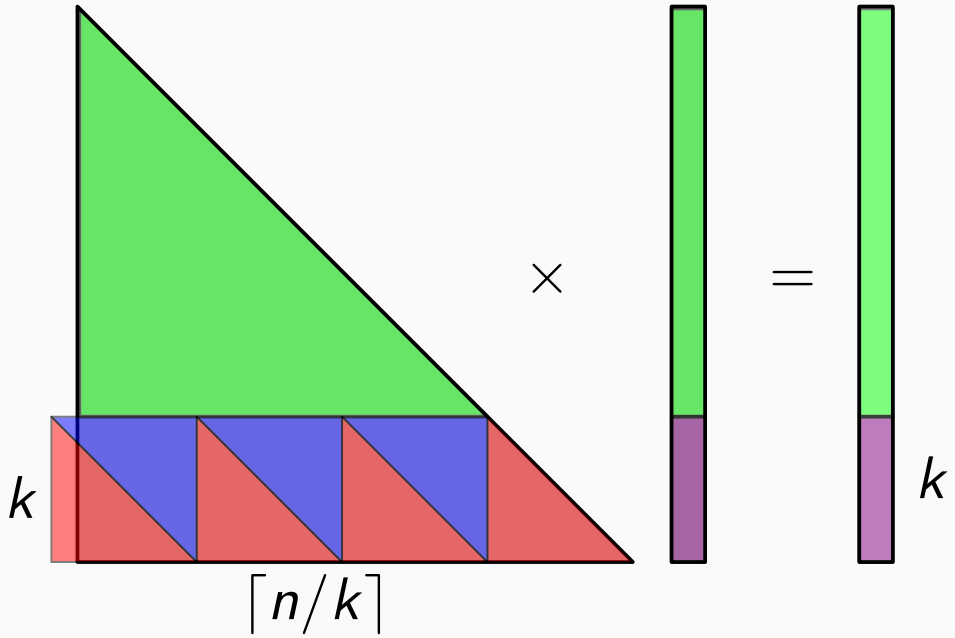

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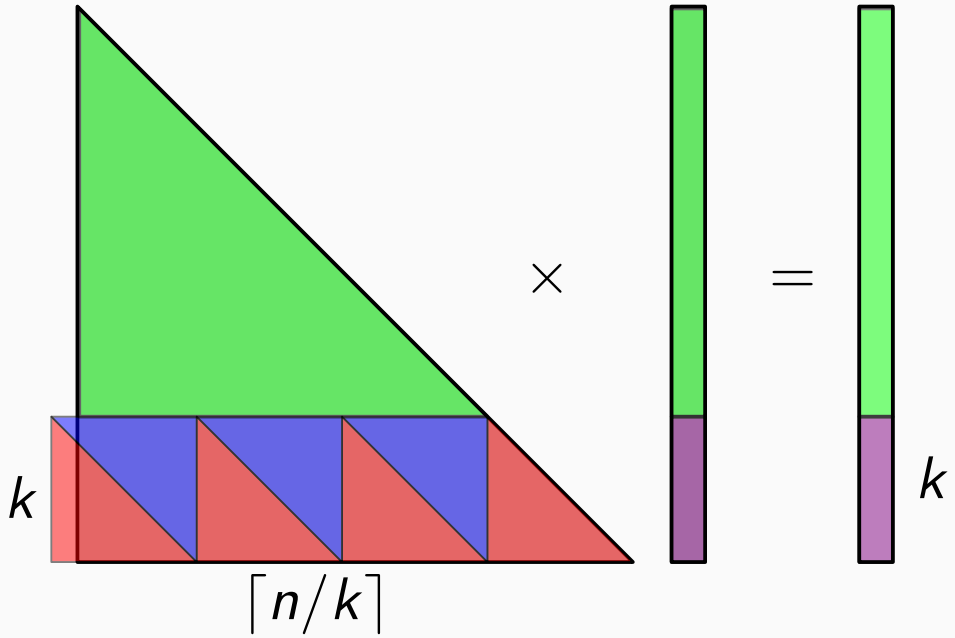

$$T(n) \leq (2c + 5)\mathrm{M}(n) + o(\mathrm{M}(n))$$

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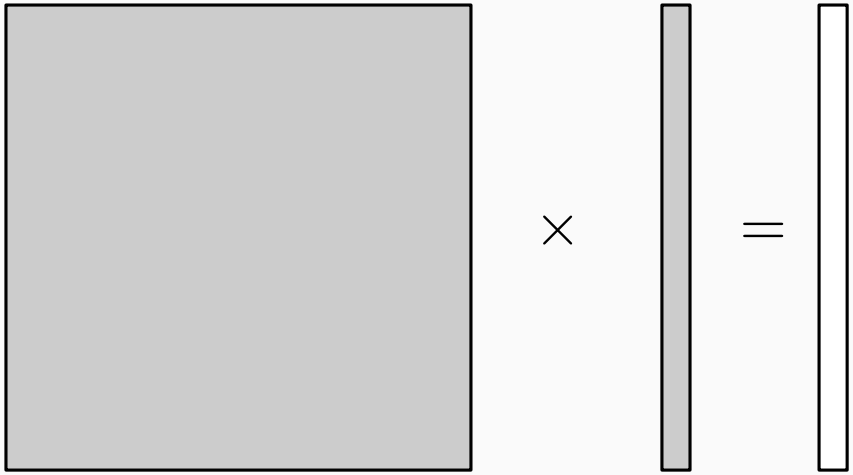

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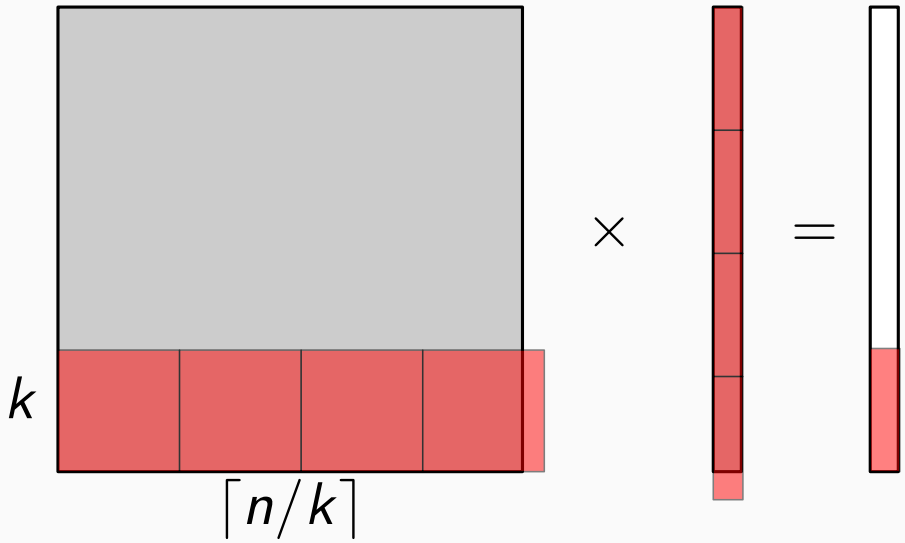

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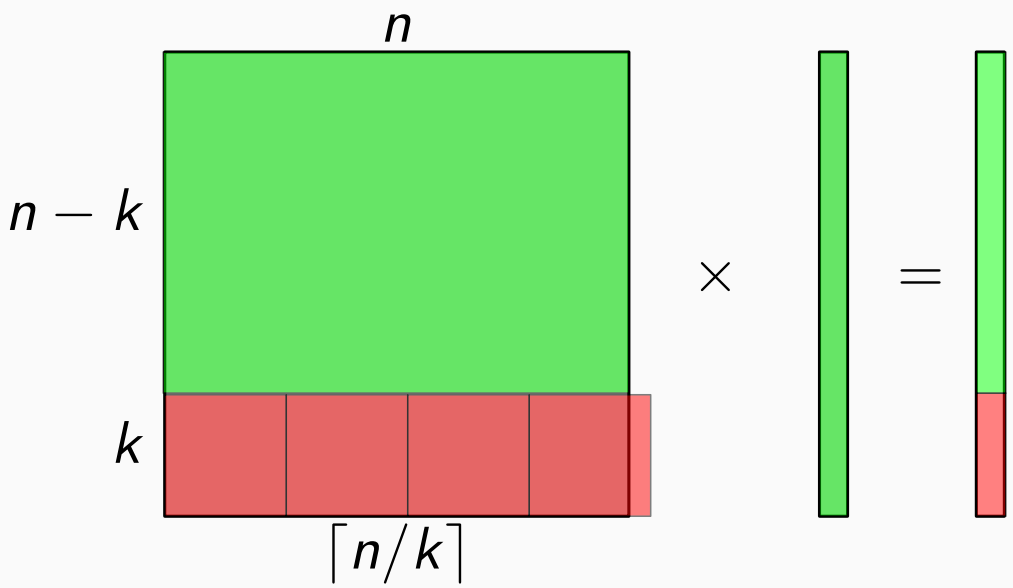

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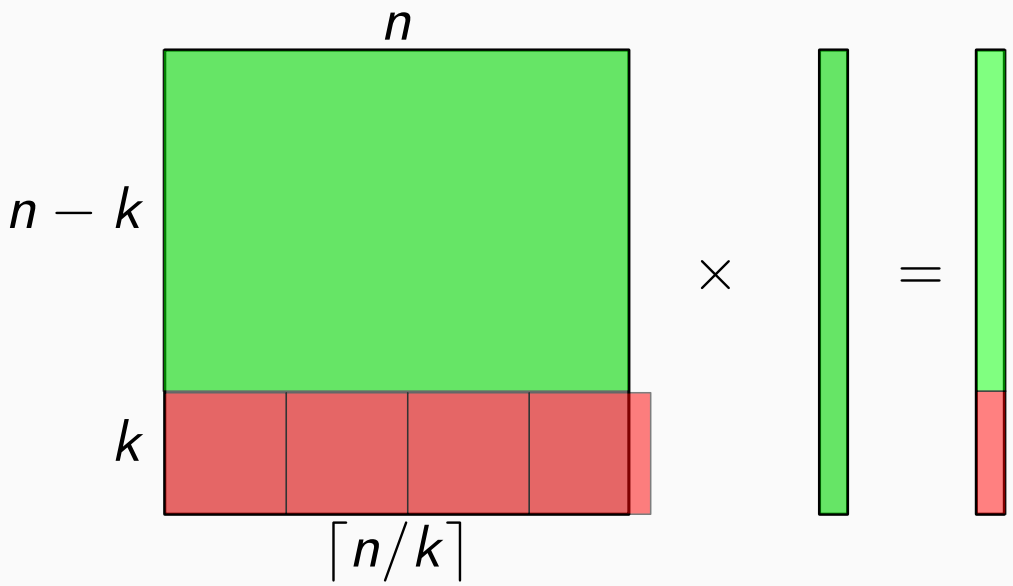

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- $T(n, m) = \lceil n/k \rceil \mathrm{M}(k) + (\lceil n/k \rceil - 1)k + T(n, m - k)$

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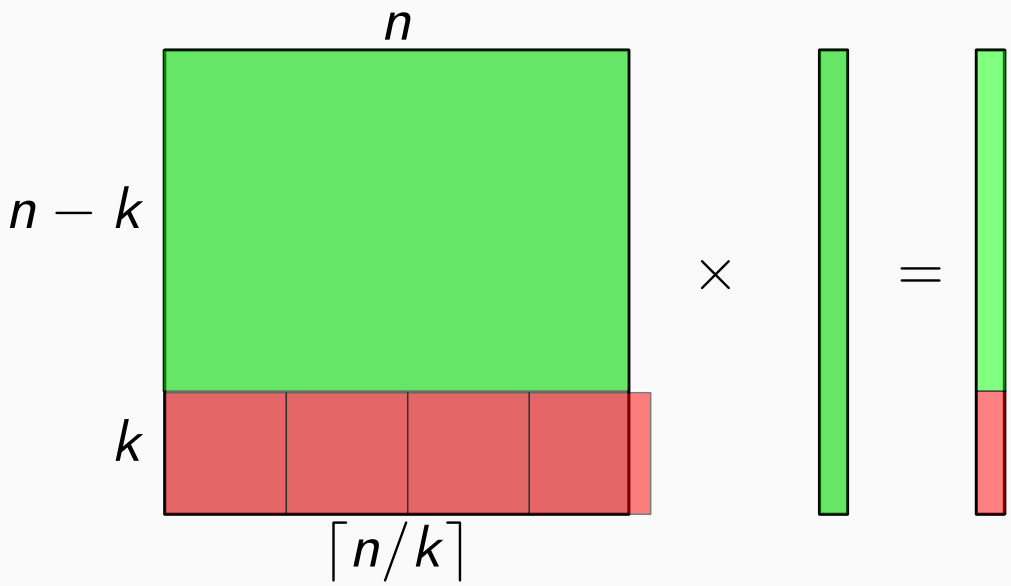

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$$T(n,n) \leq \begin{cases} \mathrm{M}(n) \log_{\frac{c+2}{c+1}}(n) + o(\mathrm{M}(n) \log n) & \text{if } \mathrm{M}(n) \text{ is quasi-linear} \\ O(\mathrm{M}(n)) & \text{otherwise} \end{cases}$$

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Thank you!