Lacunaryx: *Computing bounded-degree factors of lacunary polynomials*

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Factorization of a polynomial f

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$$= (X + Y - 1) \times (XY - 1) \times (1 + XY + \cdots + X^{568740}Y^{568740})$$

Definition

$$f(X_1, \dots, X_n) = \sum_{j=1}^{k} c_j X_1^{\alpha_{1j}} \cdots X_n^{\alpha_{nj}}$$

- $\mathrm{size}(f) \simeq k\left(\max_j(\mathrm{size}(c_j)) + n\log(\deg f)\right)$

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- **integer roots** of $f \in \mathbb{Z}[X]$; [Cucker-Koiran-Smale'98]
- **low-degree** factors of $f \in \mathbb{Q}(\alpha)[X]$; [H. Lenstra'99]
- **low-degree** factors of $f \in \mathbb{Q}(\alpha)[X_1,\ldots,X_n]$. [Kaltofen-Koiran'05-06]

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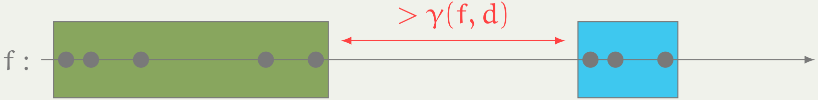

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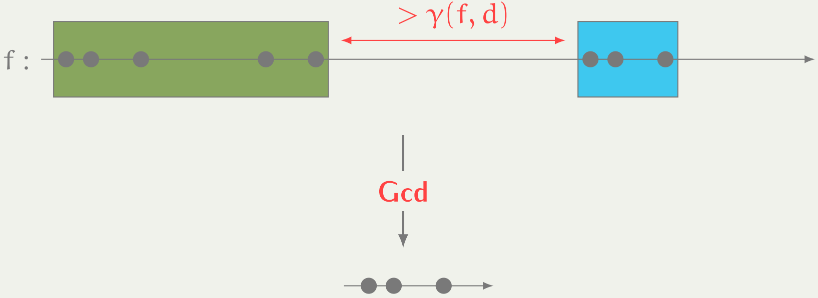

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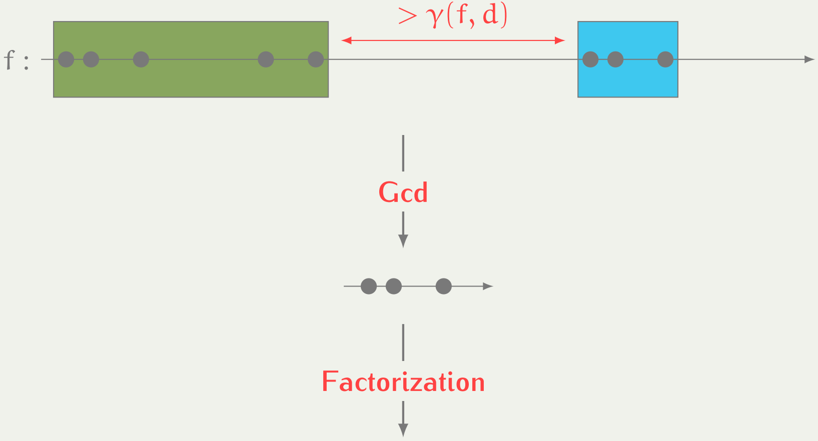

Degree-d factors of f

Modified Lenstra's algorithm

f :

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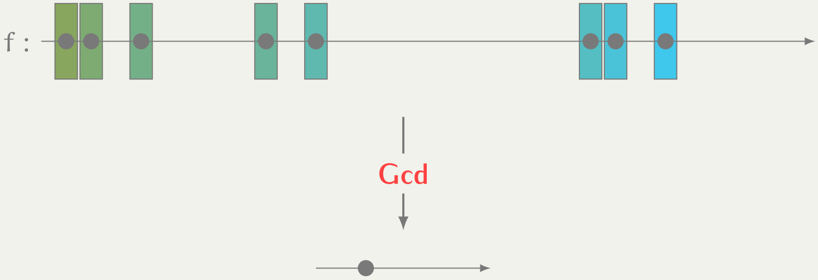

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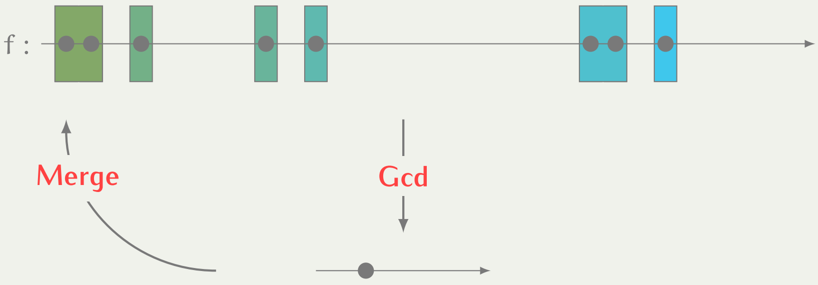

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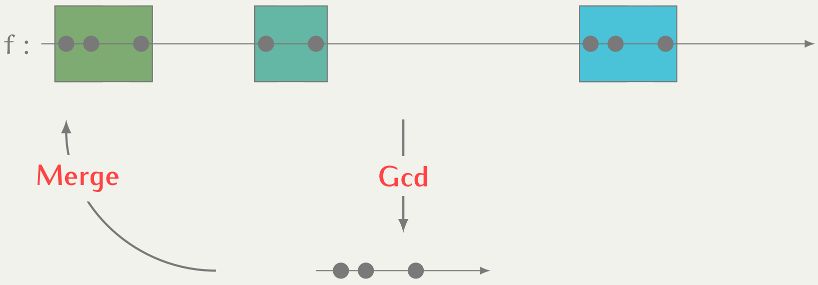

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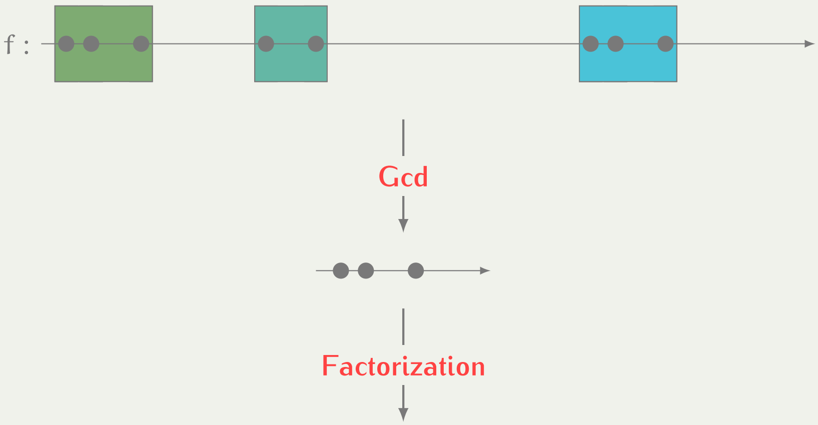

Some degree-d factors of f

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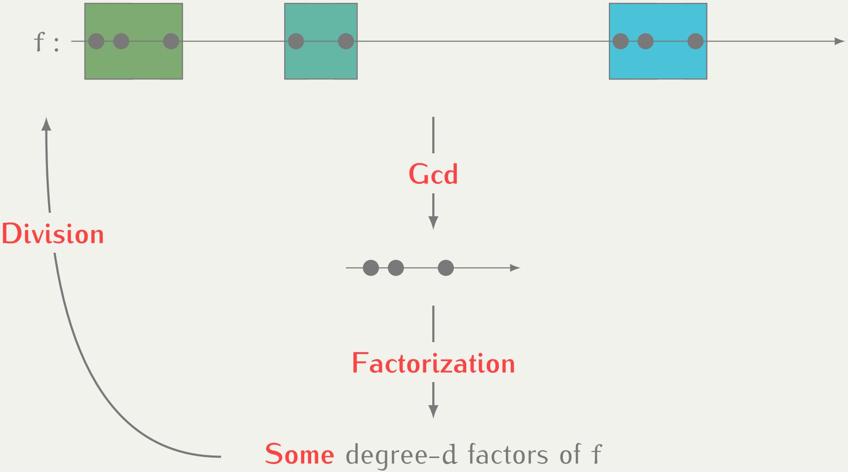

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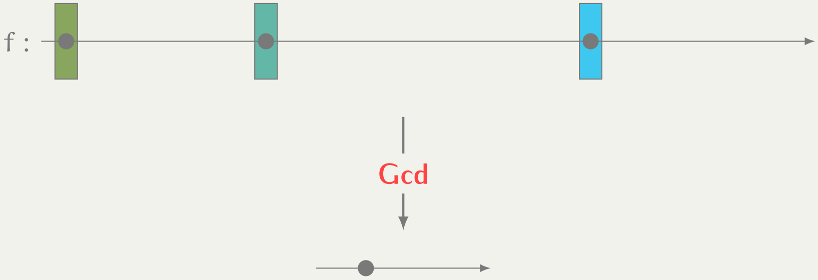

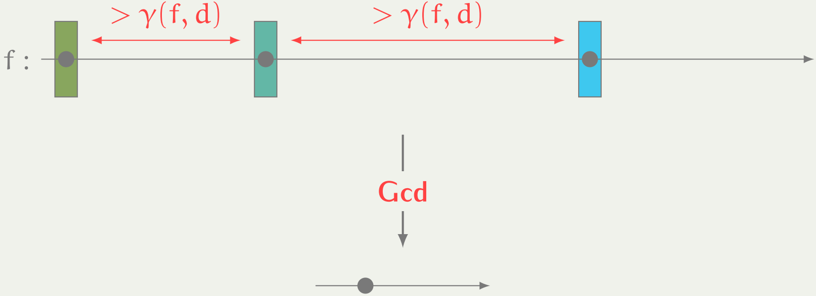
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Gcd

- Cyclotomic factors: test if each ϕ_r of degree $\leqslant d$ divides f:

$$\phi_r \text{ divides } f \iff \phi_r \text{ divides } f^{\bmod r} = \sum_j c_j X^{\alpha_j \bmod r}$$

- Multivariate polynomials: Same modifications

Implementation

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- Implementation in MATHEMAGIX as a package named LACUNARYX
  - `http://mathemagix.org > Packages > Lacunaryx`
  - Polynomials over $\mathbb{Z}$ and $\mathbb{Q}$
  - Partially glued inside MATHEMAGIX

$$f = l \times c \times s$$

- l: product of low-degree polynomials
- c: product of $X^r - 1$
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Example

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- l: product of low-degree polynomials

  5 polynomials, degree $\leqslant 10$, coefficients $\leqslant 50$

- c: product of $X^r - 1$

  3 polynomials, $r \leqslant 100\,000$

- s: *perturbated* sparse polynomial: $s = \sum_{j=1}^{n} X^{\alpha_j} p_j(X)$

  $n = 40$, $\alpha_j \leqslant 1\,000\,000$, $\deg(p_j) \leqslant 20$, coefficients $\leqslant 50$

$\rightsquigarrow$ degree $\geqslant 1\,000\,000$, $\geqslant 10\,000$ terms, coefficients $\geqslant 5 \times 10^9$

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Thank you!