Factoring bivariate lacunary polynomials without heights

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Classical factorization algorithms

Factorization of a polynomial P

Find $F_1, \ldots, F_t$, irreducible, s.t. $P = F_1 \times \cdots \times F_t$.

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$\mathbb{Q}(\alpha)[X]$

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Complexity

Polynomial in the **degree** of the polynomials

$\downarrow$

$\mathbb{Q}(\alpha)[X_1, \ldots, X_n]$
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Lacunary polynomials

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Definition

$$P(X_1, \ldots, X_n) = \sum_{j=1}^{k} a_j X_1^{\alpha_{1j}} \cdots X_n^{\alpha_{nj}}$$

- Lacunary representation: $\{(\alpha_{1j}, \ldots, \alpha_{nj} : a_j) : 1 \leqslant j \leqslant k\}$
- $\mathrm{size}(P) \simeq \sum_j \mathrm{size}(a_j) + \log(\alpha_{1j}) + \cdots + \log(\alpha_{nj})$

Factorization of lacunary polynomials

Theorems

Deterministic polynomial time (in $\log(\deg P)$) algorithms for:

- **linear** factors of **univariate** polynomials over $\mathbb{Z}$;
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  [Kaltofen-Koiran'05]
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  [Kaltofen-Koiran'06]

Integral roots of integral polynomials

Gap Theorem [Cucker-Koiran-Smale'98]

Let
$$P(X) = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j}}_{Q} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j}}_{R} \in \mathbb{Z}[X]$$

with $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k$. Suppose that

$$\alpha_{\ell+1} - \alpha_\ell > 1 + \log\left(\max_{j\leqslant\ell} |a_j|\right),$$

then for all $x \in \mathbb{Z}$, $|x| \geqslant 2$, $P(x) = 0 \implies Q(x) = R(x) = 0$.

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$$(-9 + X^2) \searrow -3 \swarrow (6 + 2X)$$

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-3 + check 0, 1 and -1

Linear factors of bivariate polynomials

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with $uv \neq 0$, $\alpha_1 \leqslant \cdots \leqslant \alpha_k$. If ℓ is the smallest index s.t.

$$\alpha_{\ell+1} > \alpha_1 + \binom{\ell}{2},$$

then $P \equiv 0$ iff both $Q \equiv 0$ and $R \equiv 0$.

Proof of the Gap Theorem

$\mathbb{K}$: any field of characteristic 0

Bound on the valuation

Definition

$\mathrm{val}(P) =$ degree of the **lowest degree monomial** of $P \in \mathbb{K}[X]$

- $\mathrm{val}(X^3 + 2X^5 - X^{17}) = 3$

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Then

$$\mathrm{val}(P) \leqslant \max_{1 \leqslant j \leqslant \ell} \left(\alpha_j + \binom{\ell + 1 - j}{2} \right).$$

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- $X^{\alpha_j}(uX+v)^{\beta_j}$ linearly independent
- If $\alpha_1 = \cdots = \alpha_\ell$, $\mathrm{val}(P) \leqslant \alpha_1 + (\ell - 1)$ [Hajós'53]

The Wronskian

Definition

Let $f_1, \ldots, f_\ell \in \mathbb{K}[X]$. Then

$$\mathrm{wr}(f_1, \ldots, f_\ell) = \det \begin{bmatrix} f_1 & f_2 & \ldots & f_\ell \\ f_1' & f_2' & \ldots & f_\ell' \\ \vdots & \vdots & & \vdots \\ f_1^{(\ell-1)} & f_2^{(\ell-1)} & \ldots & f_\ell^{(\ell-1)} \end{bmatrix}.$$

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Proposition [Bôcher, 1900]

$\mathrm{wr}(f_1, \ldots, f_\ell) \neq 0 \iff$ the f_j's are linearly independent.

Wronskian & valuation

Lemma

$$\mathrm{val}(\mathrm{wr}(f_1, \dots, f_\ell)) \geqslant \sum_{j=1}^{\ell} \mathrm{val}(f_j) - \binom{\ell}{2}$$

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$$\sum_{j=1}^{\ell} \alpha_j \geqslant \mathrm{val}(\mathrm{wr}(f_1,\dots,f_\ell)) \geqslant \mathrm{val}(P) + \sum_{j=2}^{\ell} \alpha_j - \binom{\ell}{2}$$

How far from optimality?

$$\mathrm{val}\left(\sum_{j=1}^{\ell} a_j X^{\alpha_j}(uX+v)^{\beta_j}\right) \leqslant \begin{cases} \alpha_1 + (\ell - 1) & \text{[Hajós'53] (constant } \alpha_j) \\ \alpha_1 + \binom{\ell}{2} & \text{[Our result]} \end{cases}$$

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- Lemmas: tight, but not simultaneously
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$$P_\ell(X) = (1+X)^{2\ell+3} - 1 - \sum_{j=3}^{\ell} \frac{2\ell-3}{2j-5}\binom{\ell+j-5}{2j-6} X^{2j-5}(1+X)^{\ell-1-j}$$
$$= X^{2\ell-3}$$

Gap Theorem

Theorem

Let

$$P = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j}(uX+v)^{\beta_j}}_{Q} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j}(uX+v)^{\beta_j}}_{R}$$

with $uv \neq 0$, $\alpha_1 \leqslant \cdots \leqslant \alpha_k$. If ℓ is the smallest index s.t.

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then $P \equiv 0$ iff both $Q \equiv 0$ and $R \equiv 0$.

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with $uv \neq 0$, $\alpha_1 \leqslant \cdots \leqslant \alpha_k$. If ℓ is the smallest index s.t.

$$\alpha_{\ell+1} > \alpha_1 + \binom{\ell}{2} \geqslant \mathrm{val}(Q),$$

then $P \equiv 0$ iff both $Q \equiv 0$ and $R \equiv 0$.

$$P = \left(c_{\mathrm{val}(Q)} X^{\mathrm{val}(Q)} + \cdots\right) + X^{\alpha_{\ell+1}} \left(a_{\ell+1} (uX+v)^{\beta_{\ell+1}} + \cdots\right)$$

Algorithms

$\mathbb{K} = \mathbb{Q}(\alpha)$: **algebraic number field**

Finding linear factors

Observation + Gap Theorem (recursively)

$(Y - uX - v)$ divides $P(X, Y)$

$$\iff P(X, uX + v) \equiv 0$$

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$$\iff (Y - uX - v) \text{ divides each } P_t(X,Y)$$

Finding linear factors

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$$
\begin{aligned}
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- Independent from u and v
- X does not play a special role

Example

$$P = X^{31}Y^6 - 2X^{30}Y^7 + X^{29}Y^8 - X^{29}Y^6 + X^{18}Y^{13} - X^{16}Y^{15} + X^{17}Y^{13} + X^{16}Y^{14} + X^{10}Y^2 - X^9Y^3 + X^9Y^2 - X^5Y^6 + X^3Y^8 - 2X^3Y^7 + X^3Y^6$$

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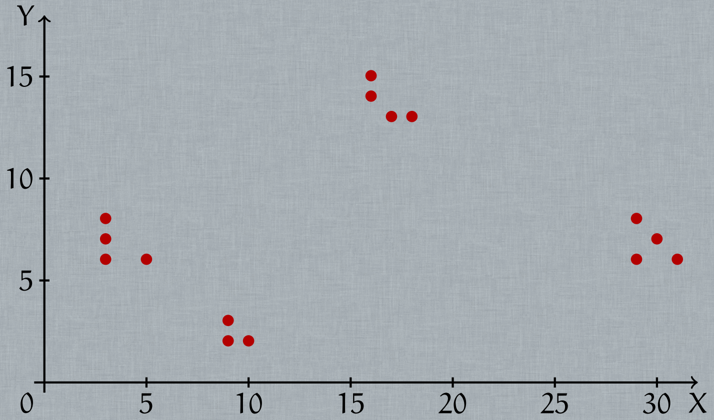
$$+ X^9Y^2 - X^5Y^6 + X^3Y^8 - 2X^3Y^7 + X^3Y^6$$



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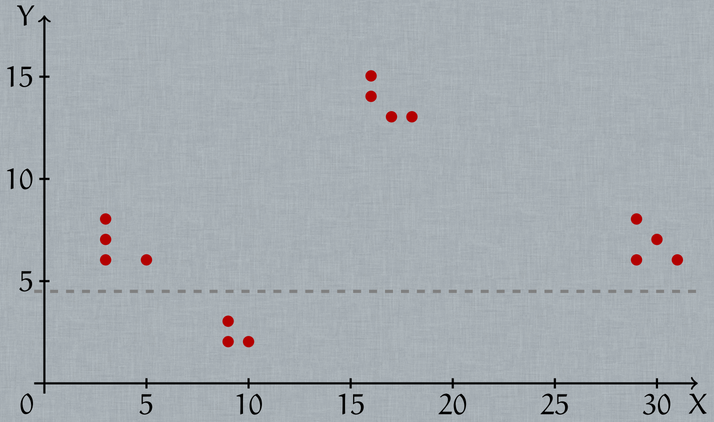

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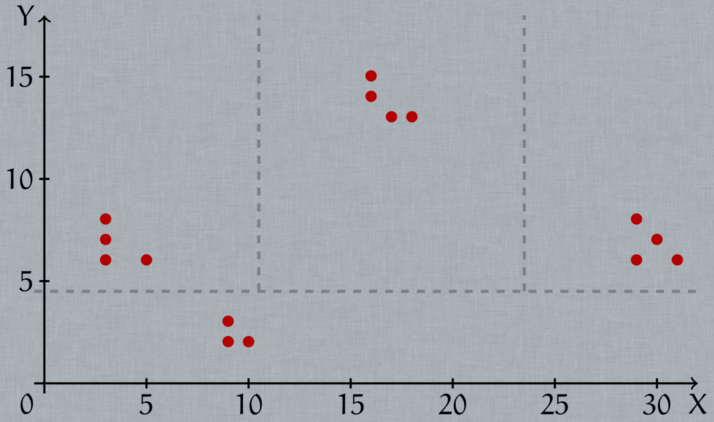

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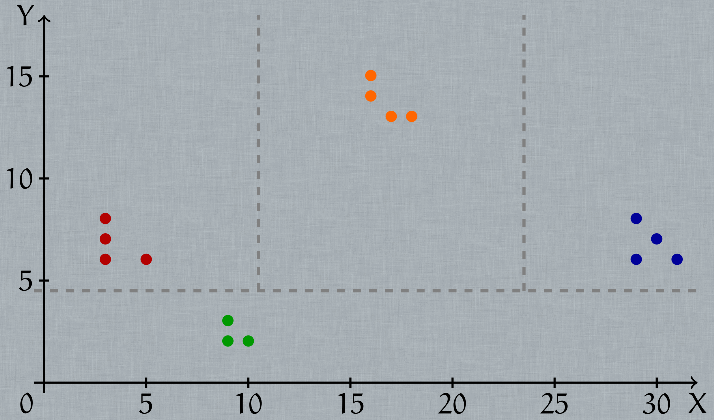

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$$P = X^{31}Y^6 - 2X^{30}Y^7 + X^{29}Y^8 - X^{29}Y^6 + X^{18}Y^{13} - X^{16}Y^{15} + X^{17}Y^{13} + X^{16}Y^{14} + X^{10}Y^2 - X^9Y^3 + X^9Y^2 - X^5Y^6 + X^3Y^8 - 2X^3Y^7 + X^3Y^6$$

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$\Longrightarrow$ linear factors of P: $(X - Y + 1, 1)$, $(X, 3)$, $(Y, 2)$

Complete algorithm

Find linear factors of $P(X,Y) = \sum_{j=1}^{k} a_j X^{\alpha_j} Y^{\beta_j}$

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Univariate lacunary factorization

[H. Lenstra'99]

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monomials — binomials — trinomials

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$(\deg(P_t) \leqslant \mathcal{O}(\ell_t^2))$

Low-degree factorization
[Kaltofen'82, ..., Lecerf'07]

Complete algorithm

Let $P = \sum_{j=1}^{k} a_j X^{\alpha_j} Y^{\beta_j} \in \mathbb{Q}(\alpha)[X,Y]$ be given in lacunary representation. There exists a **deterministic polynomial-time** algorithm to compute its linear factors, with multiplicities.

monomials binomials trinomials

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Positive characteristic

$\mathbb{K} = \mathbb{F}_{p^s}$: **field with p^s elements**

Valuation & PIT

$$(1+X)^{2^n} + (1+X)^{2^{n+1}} = X^{2^n}(X+1) \mod 2$$

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monomials | binomials | trinomials

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$(X - a)$ …

Factors …

… $X)$

R… $u \mapsto \sum_j a_j u^{\beta_j}$

NP-complete
under BPP reductions

[Kipnis-Shamir'99,Bi-Cheng-Rojas'13]

Talk at 2:25pm

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