Symmetric Determinantal Representations of Polynomials

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The problem

$$(x+3y)z = \det\begin{pmatrix} 0 & x & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & z \\ 0 & 0 & 1 & y & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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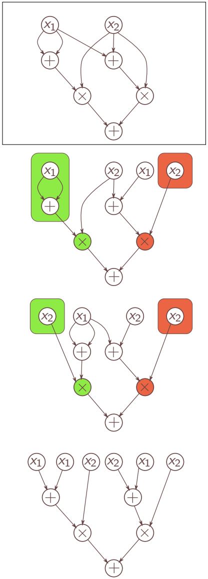
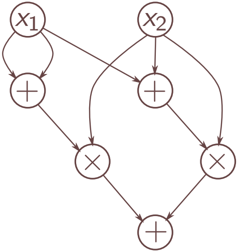
- Formal polynomial
- Smallest possible dimension of the matrix

Representations of polynomials





Circuit of size 5

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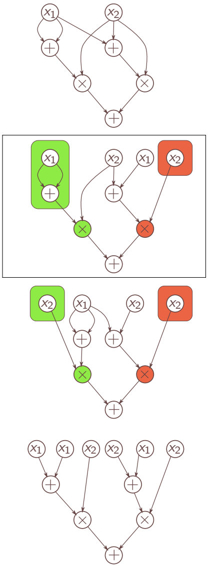

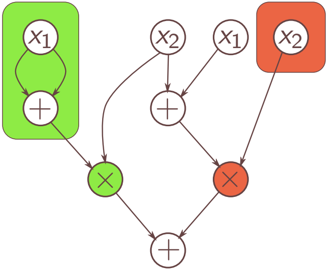

Weakly-skew circuit of size 5

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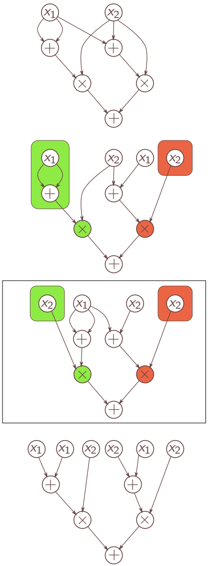

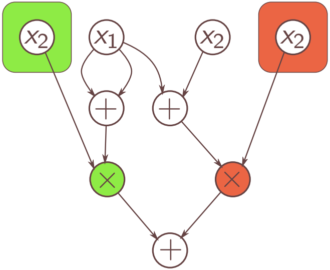

Skew circuit of size 5

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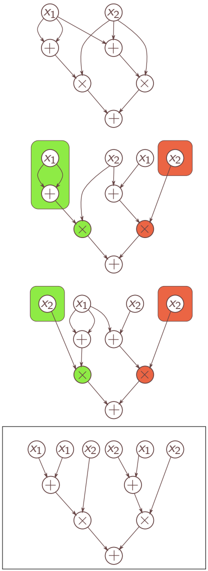
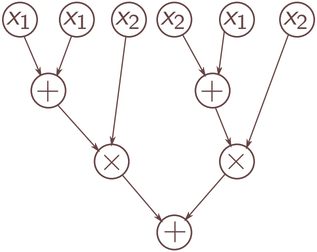
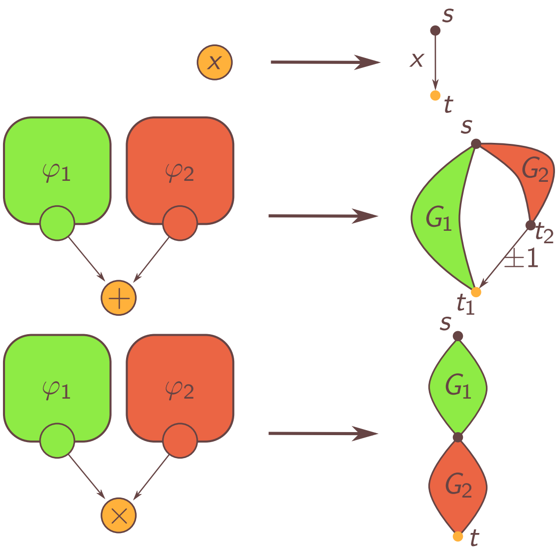




Formula of size 5

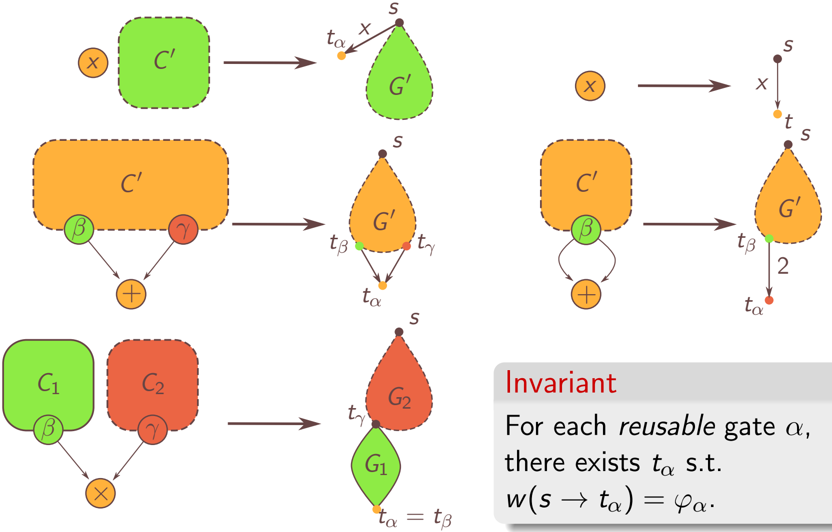
Valiant's construction



Invariant

$$\varphi = \pm \sum_{s\text{-}t\text{-paths } P} (-1)^{|P|} w(P)$$

Toda-Malod's construction



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For each *reusable* gate α, there exists t_α s.t. $w(s \to t_\alpha) = \varphi_\alpha$.

Motivation from Convex Geometry

- Linear Matrix Expression (LME): for A_i symmetric in $\mathbb{R}^{t\times t}$

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- Drop condition $A_0 \succeq 0$ $\rightsquigarrow$ exponential size matrices
- What about polynomial size matrices?

Outline

1. Universality of determinants of symmetric matrices

2. Characteristic 2

Introduction

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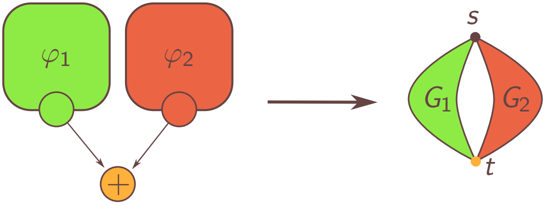
- N.B.: $\mathrm{char}(\mathbb{K}) \neq 2$ in this section

From formulas to symmetric determinants



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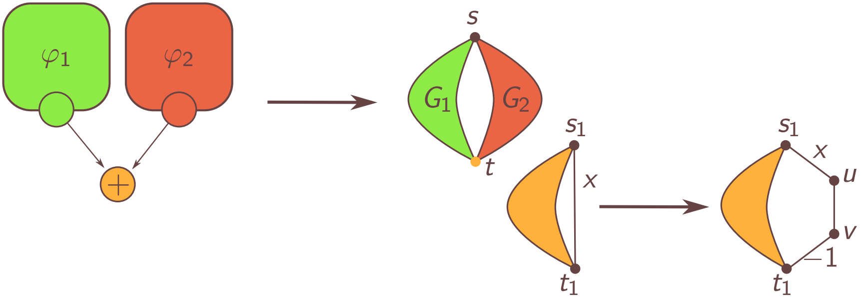

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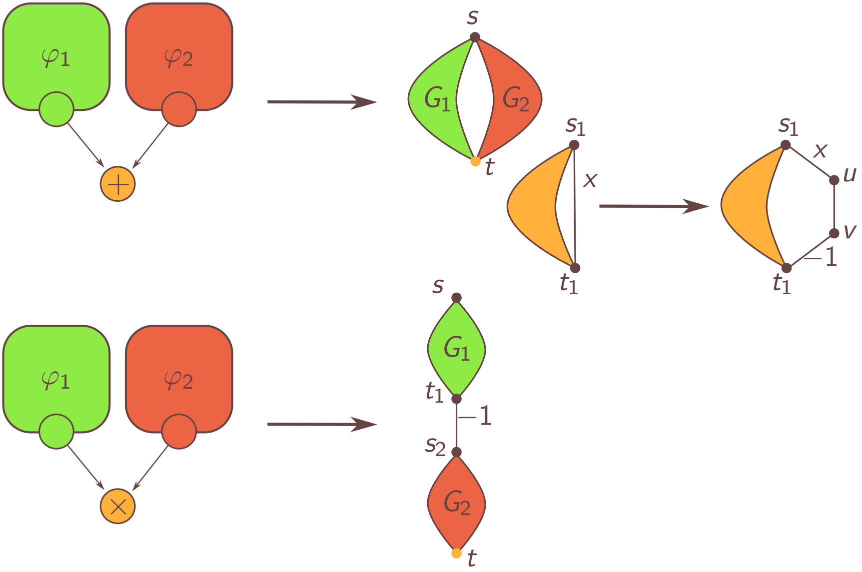

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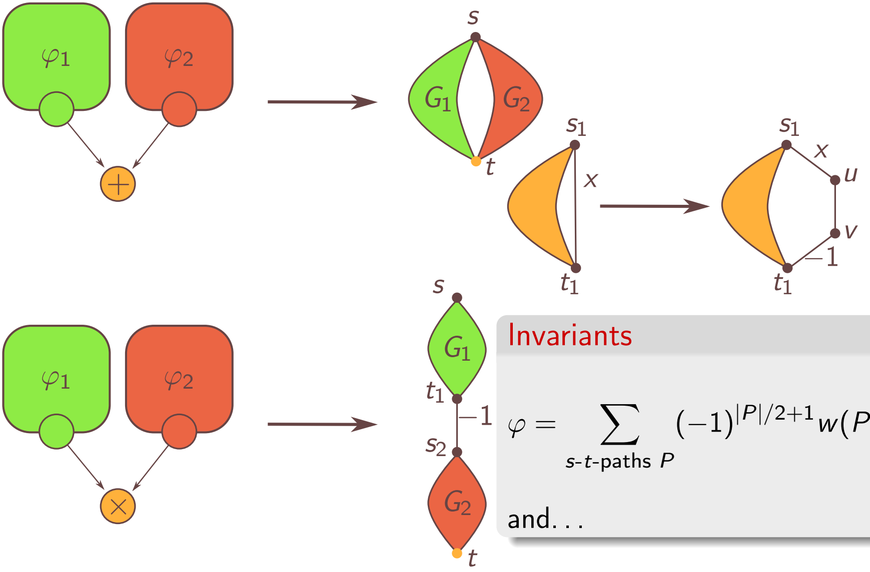

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and...

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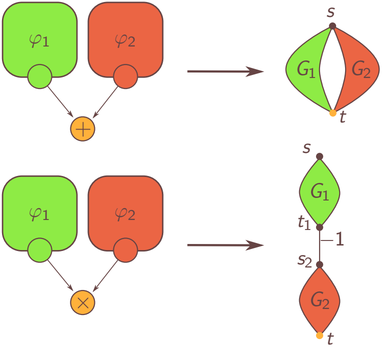

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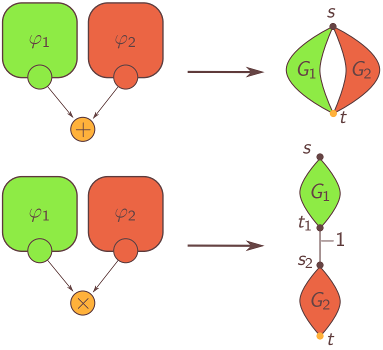

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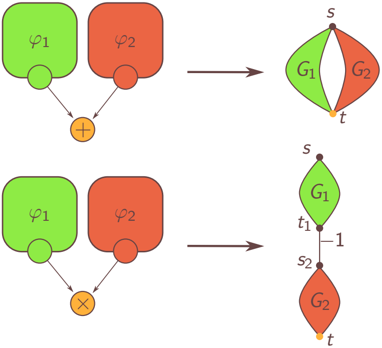

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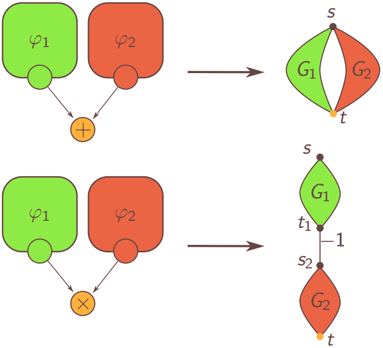

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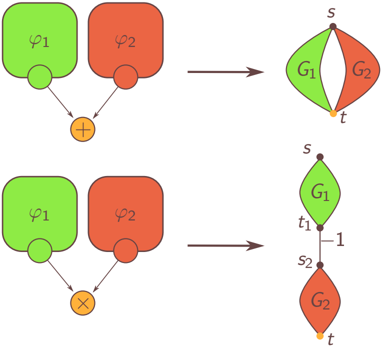

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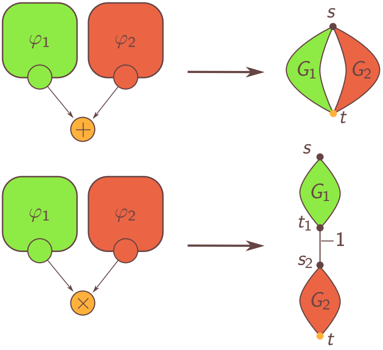
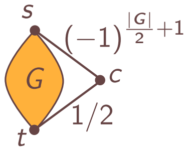


From G to G'



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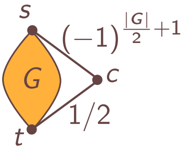

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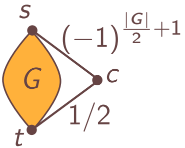

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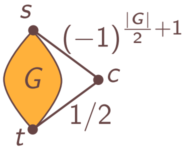

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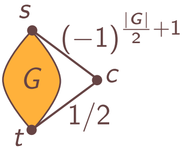

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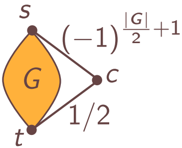

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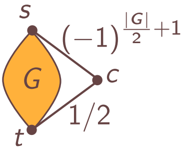

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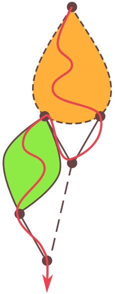
Theorem

For a formula φ of size e, this construction yields a graph of size $2e + 3$. The determinant of its adjacency matrix equals φ.

Case of weakly-skew circuits

- Main difficulty:



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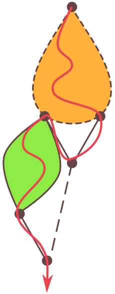
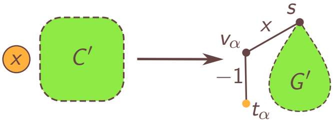


- Definition: A path P is acceptable if $G \setminus P$ admits a cycle cover

Constructions



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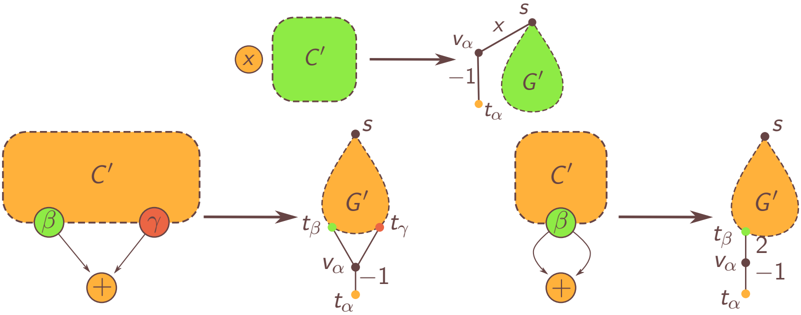

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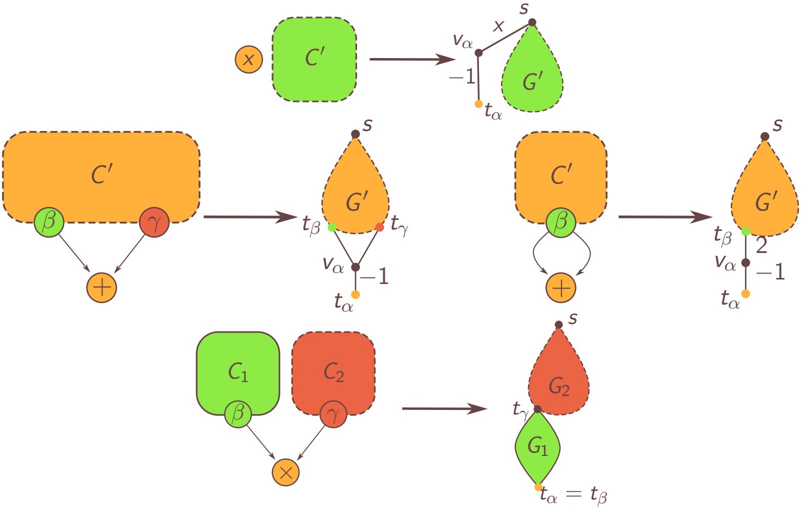

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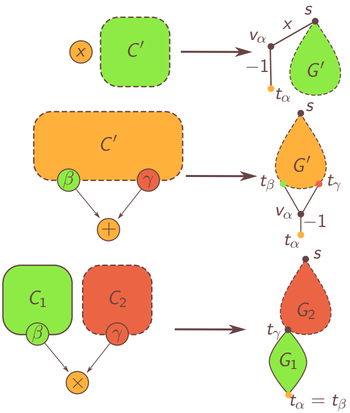

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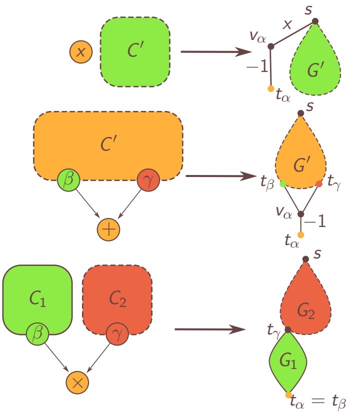

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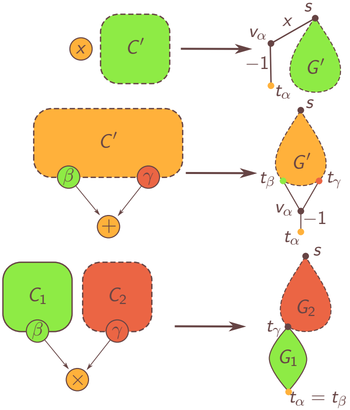

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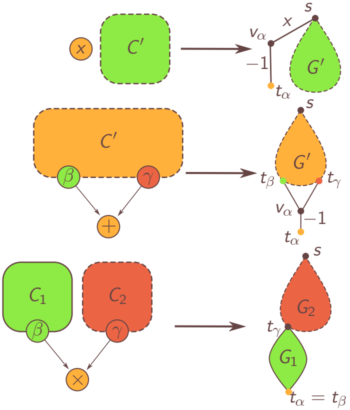

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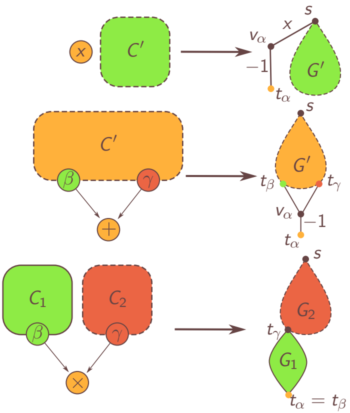

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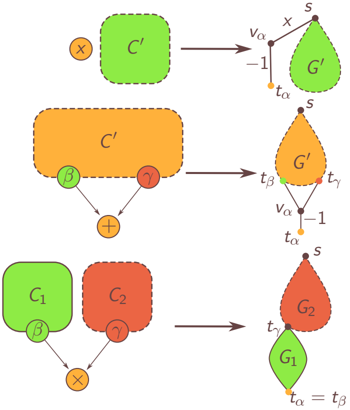

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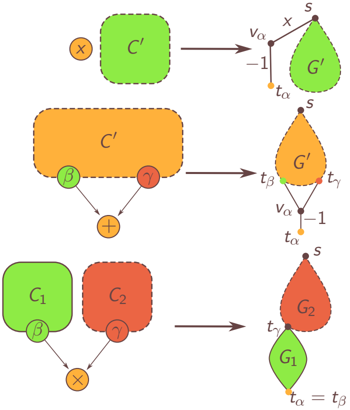

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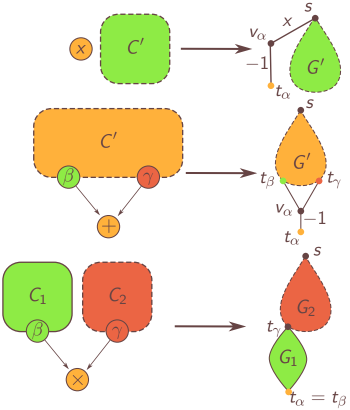

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Theorem

For a weakly skew circuit of size e, with i input variables, computing a polynomial φ, this construction yields a graph G' with $2(e+i)+1$ vertices. The adjacency matrix of G' has its determinant equal to φ.

Summary

	Formula	Weakly-skew circuit
Non symmetric	$e+1$	$(e+i)+1$
Symmetric	$2e+1$	$2(e+i)+1$

e: size

i: number of input variables

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- Here: Polynomials over $\mathbb{F}[x, y, z]$

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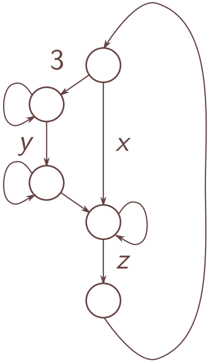

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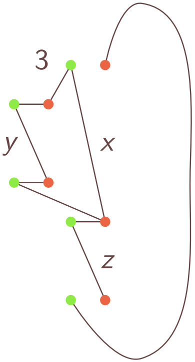

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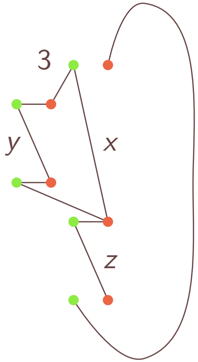

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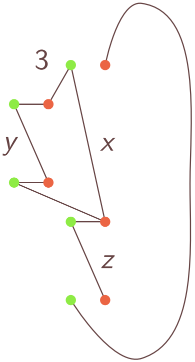

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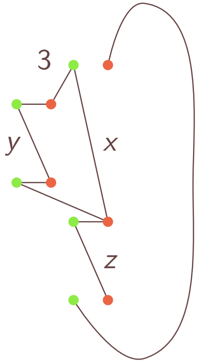

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Theorem (G., Monteil, Thomassé)

If there exists a symmetric matrix A such that $p = \det A$, then $p \bmod \langle x^2 + \ell_x, y^2 + \ell_y, z^2 + \ell_z \rangle$ is a product of degree-1 polynomials.

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- It appears to be related to an open problem of Bürgisser:

Is the partial permanent VNP-complete in characteristic 2?

Valiant's classes

- Complexity of a polynomial: size of the smallest circuit computing it.

Definition

A family (f_n) of polynomials is in VP if for all n, the number of variables, the degree, and the complexity of f_n are polynomially bounded in n.

A family (f_n) of polynomials is in VNP if there exists a family $(g_n(y_1, \dots, y_{v(n)})) \in \mathrm{VP}$ s.t.

$$f_n(x_1, \dots, x_{u(n)}) = \sum_{\bar{\epsilon} \in \{0,1\}^{v(n)-u(n)}} g_n(x_1, \dots, x_{u(n)}, \bar{\epsilon}).$$

- $(\mathrm{DET}_n) \in \mathrm{VP}$, $(\mathrm{PER}_n) \in \mathrm{VNP}$, ...

VNP-completeness

Definition

A family (g_n) is a p-projection of a family (f_n) is there exists a polynomial t s.t. for all n, $g_n(\bar{x}) = f_{t(n)}(a_1, \dots, a_m)$, with $a_1, \dots, a_m \in \mathbb{K} \cup \{x_1, \dots, x_n\}$.

A family $(f_n) \in$ VNP is VNP-complete if every family in VNP is a p-projection of (f_n).

- (PER_n) is VNP-complete in characteristic $\neq 2$
- (HC_n) is VNP-complete (in any characteristic)

Boolean parts

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The boolean part of (f_n) is $bp_f : \{0,1\}^\star \to \{0,1\}$ s.t. for $x \in \{0,1\}^n$, $bp_f(x) = f_n(x)$.

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Theorem (Bürgisser)

- $BP(\mathrm{VP}) \subseteq \mathrm{NC}^2/\mathrm{poly}$
- $BP(\mathrm{VNP}) = \oplus\mathrm{P}/\mathrm{poly}$

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Same kind of ideas as the previous proof.

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Thank you!