The real τ-conjecture & lower bounds for the permanent

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Rencontres CoA — 22 novembre 2012

Arithmetic Circuits

$$f(x,y,z) = x^4 + 4\,x^3y + 6\,x^2y^2 + 4\,xy^3 + x^2z + 2\,xyz + y^2z + x^2 + y^4 + 2\,xy + y^2 + z^2 + 2\,z + 1$$

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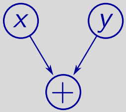

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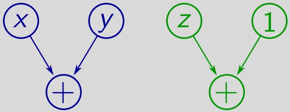

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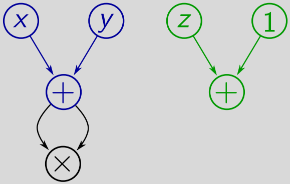

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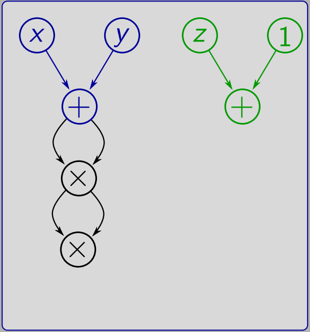

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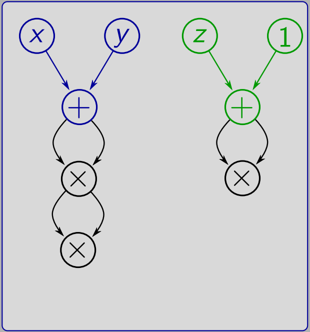

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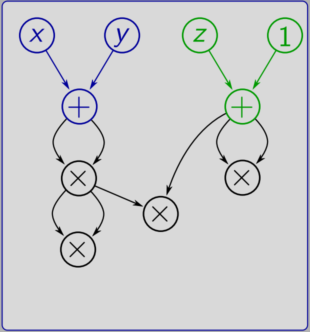

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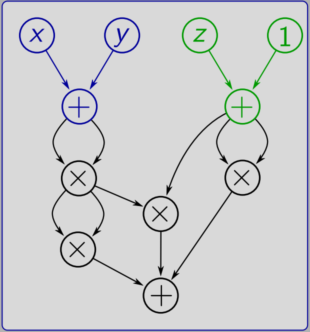

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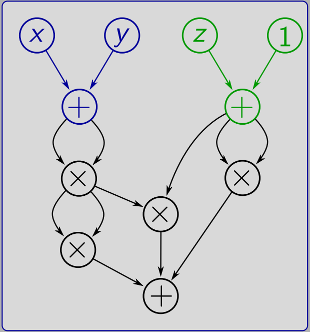

Complexity of a polynomial

$\tau(f) =$ size of its smallest circuit representation

The τ-conjecture

Conjecture (Shub & Smale, 1995)

The number of integer roots of any $f \in \mathbb{Z}[X]$ is $\leq \text{poly}(\tau(f))$.

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$$\mathrm{PER}_n(x_{11}, \ldots, x_{nn}) = \mathrm{per}\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} x_{i\sigma(i)}$$

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Extended τ-conjecture $\implies$ *Merel torsion theorem, ...*

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False for real roots (Shub-Smale 95, Borodin-Cook 76)

$T_n = n$-th Chebyshev polynomial

- $\tau(T_n) = \mathcal{O}(\log n)$
- n real roots

Let's make it real!

Real τ-conjecture (Koiran, 2011)

Let $f = \sum_{i=1}^{k} \prod_{j=1}^{m} f_{ij}$ where the f_{ij}'s are t-sparse polynomials.

Then f has $\leq \text{poly}(k, m, t)$ real roots.

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- Toy question: Number of real roots of $fg + 1$?

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Theorem

If $f \in \mathbb{R}[X]$ has t monomials, then it has $\leq (t-1)$ positive real roots.

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$f = \sum_{i=1}^{k} \prod_{j=1}^{m} f_{ij}$: $\leq 2kt^m - 1$ real roots

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$+$ other details…

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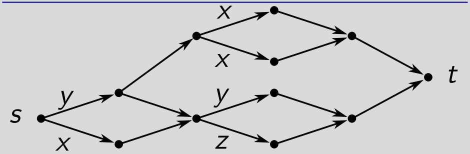

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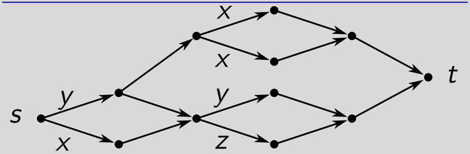

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Consequence. Replace $\mathrm{poly}(k, m, t)$ by $2^{\mathrm{polylog}(k,m,t)}$.

The limited power of powering

$$\mathrm{SPS}(k, m, t, A) = \left\{ \sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}} : f_j\text{'s are } t\text{-sparse}, \ \alpha_{ij} \leq A \right\}$$

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Theorem (G.-Koiran-Portier-Strozecki, 2011)

If $f \in \mathrm{SPS}(k,m,t,A)$, its number of real roots is at most

$$C \cdot \left[e \cdot \left(1 + \frac{t^m}{2^{k-1}-1} \right) \right]^{2^{k-1}-1} \quad \textit{for some } C.$$

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- If k and m are fixed, this is polynomial in t.

Case $k = 2$

Proposition

The polynomial

$$f = \prod_{j=1}^{m} f_j^{\alpha_j} + \prod_{j=1}^{m} f_j^{\beta_j}$$

has at most $2mt^m + 4m(t-1)$ *real roots.*

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$$F' = \underbrace{\prod_{j=1}^{m} f_j^{\beta_j - \alpha_j - 1}}_{\leq 2m(t-1) \text{ roots and poles}} \times \underbrace{\sum_{j=1}^{m} (\beta_j - \alpha_j) f_j' \prod_{l \neq j} f_l}_{\leq 2mt^m - 1 \text{ roots}} .$$

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Let f, g be t-sparse polynomials.

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