Symmetric Determinantal Representations of Polynomials

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The problem

$$(x+y)+(y\times z) = \det \begin{vmatrix} 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ x & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{vmatrix}$$

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- Formal polynomial

The problem

$$
(x+y)+(y\times z)=\det\begin{vmatrix}
0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\\
x & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
y & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 1 & 0 & z & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0\\
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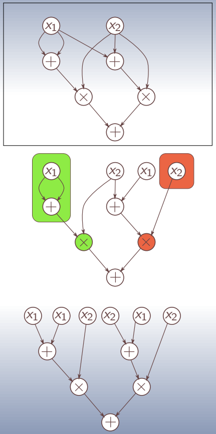
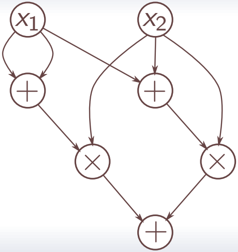
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{vmatrix}
$$

- Formal polynomial
- Smallest possible dimension of the matrix

Representations of polynomials



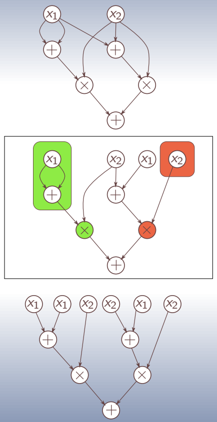
Arithmetic circuit:



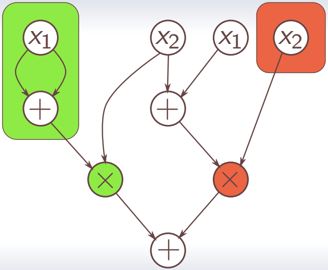
Size $e = 5$

Inputs $i = 2$

Representations of polynomials



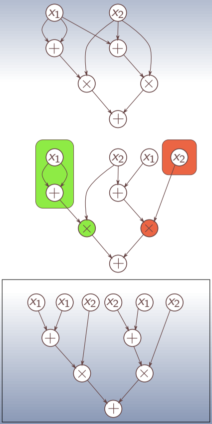
Weakly-skew circuit:



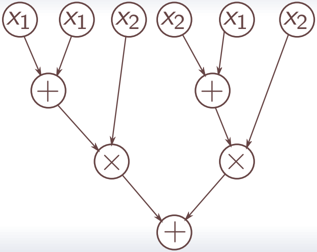
Size $e = 5$

Inputs $i = 4$

Representations of polynomials



Formula:



Size $e = 5$

Inputs $i = 6$

Motivation

L. G. Valiant, **Completeness classes in algebra**, STOC'79

Theorem (Universality of determinant and permanent)

Let P be a polynomial given by a formula of size e. There exist matrices M and N of size $(e+2)\times(e+2)$ such that

$$P = \det M = \operatorname{per} N.$$

Subsequent works

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  - G. Malod & N. Portier [4] $e+i+1$

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[3] **Classes of arithmetic circuits capturing the complexity of computing the determinant**, IEICE T. Inf. Syst., 1992.

[4] **Characterizing Valiant's algebraic complexity classes**, J. Compl., 2008.

Main results

- Extension to symmetric matrices (char. $\neq 2$)

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[1] With E. L. Kaltofen, P. Koiran, N. Portier. Symmetric Determinantal Representation of Weakly-Skew Circuits, Proc. 28th STACS, 2011.

[2] With T. Monteil, S. Thomassé. Symmetric Determinantal Representations in Characteristic 2, in preparation, 2011.

Motivation from Convex Geometry

- Linear Matrix Expression (LME): for A_i symmetric in $\mathbb{R}^{t\times t}$

$$A_0 + x_1 A_1 + \cdots + x_n A_n$$

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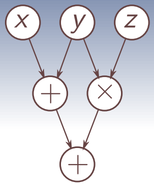
More on this: Tim Netzer's talk (Friday 9:30am @Riddick 339)

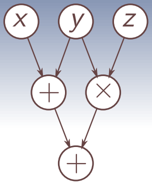
Overview

$$(x + y) + (y \times z)$$



Circuit: Weakly-skew circuit or formula

Overview



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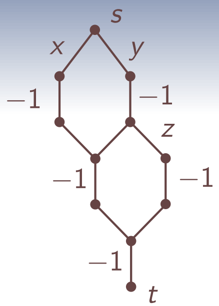
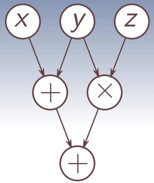
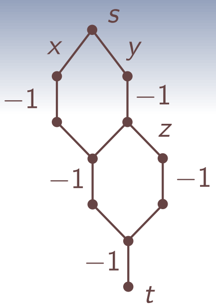




Arithmetic Branching Program

Circuit $\Longrightarrow$ ABP

Overview



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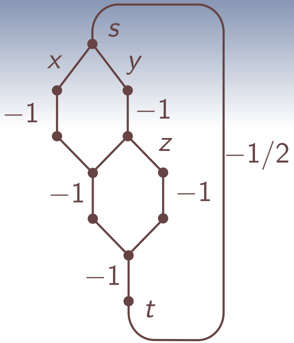
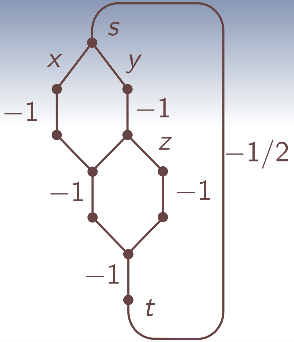


Circuit $\Longrightarrow$ ABP $\Longrightarrow$ Graph

Overview



$$\det\begin{vmatrix} 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ x & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{vmatrix}$$

$$= (x + y) + (y \times z)$$

Circuit $\Longrightarrow$ ABP $\Longrightarrow$ Graph $\Longrightarrow$ Matrix

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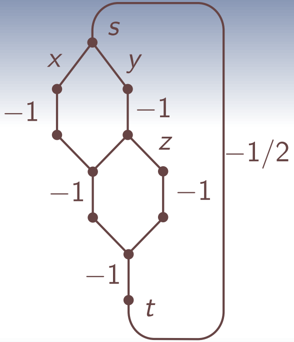

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$$= (x + y) + (y \times z)$$

Characteristic $\neq 2$

Circuit $\Longrightarrow$ ABP $\Longrightarrow$ Graph $\Longrightarrow$ Matrix

Two remarks

Symmetric matrices

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$\Longrightarrow$ undirected graphs

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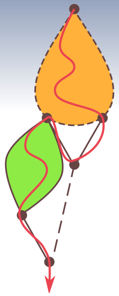

Symmetric matrices

$\Longrightarrow$ undirected graphs

$\Longrightarrow$ "undirected ABPs"

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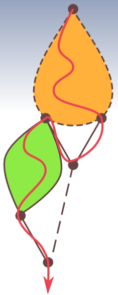
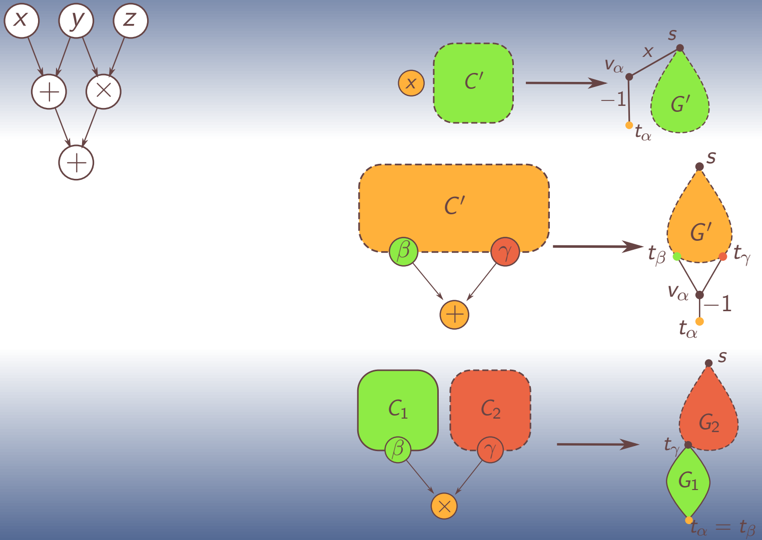


Symmetric matrices
$\Longrightarrow$ undirected graphs
$\Longrightarrow$ "undirected ABPs"

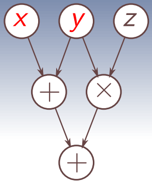
Corollary

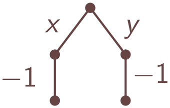
Let M be an $n \times n$ matrix. Then there exists a symmetric matrix M' of size $O(n^3)$ s.t. $\det M = \det M'$.

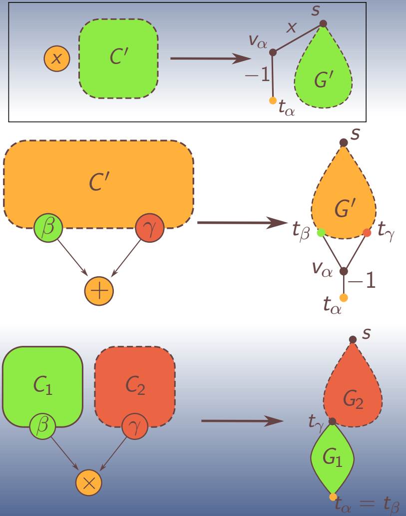
Weakly-Skew Circuit $\Longrightarrow$ ABP



Weakly-Skew Circuit $\implies$ ABP







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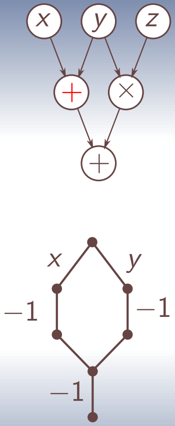

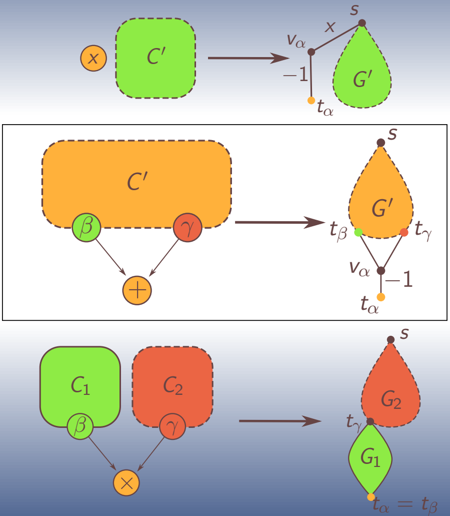

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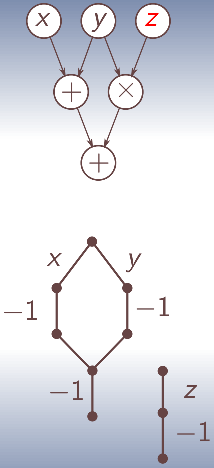

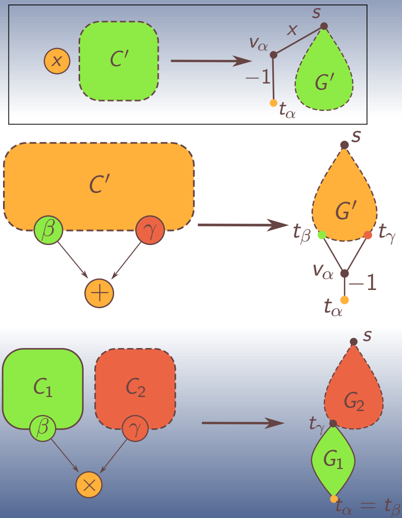

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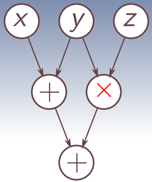

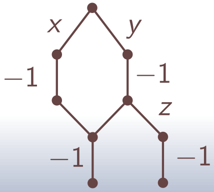

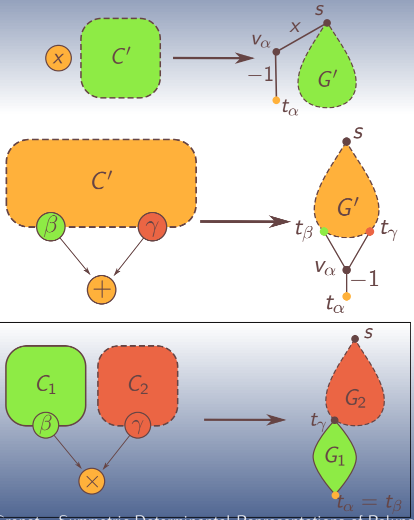

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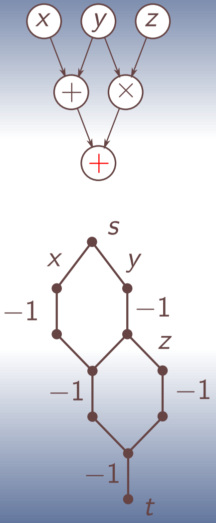
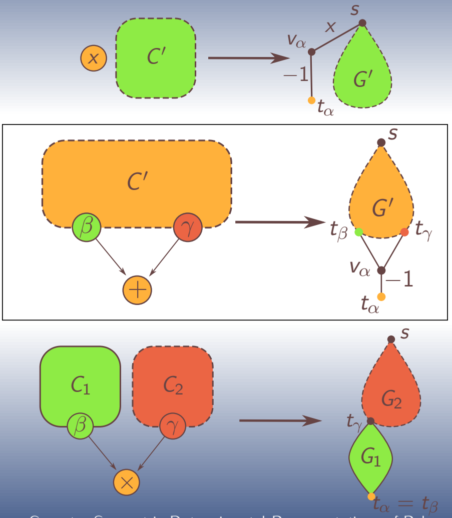
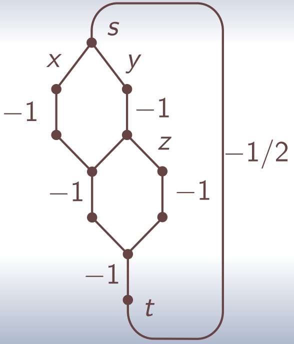


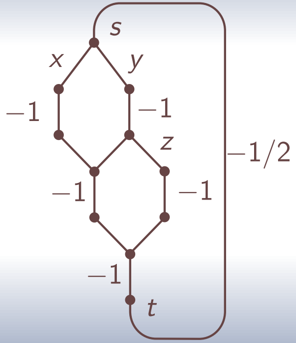


ABP $\Longrightarrow$ Graph



- Add $s \overset{(1/2)\cdot(-1)^{\frac{|G|-1}{2}}}{\longleftrightarrow} t$: new graph G'.

ABP $\implies$ Graph



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- Cycle covers of G'

  $\iff$ $s \to t$-paths in G

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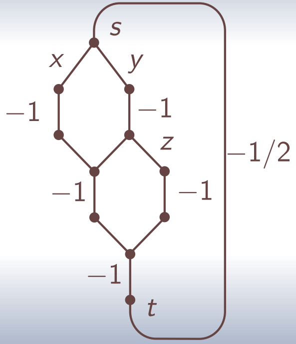

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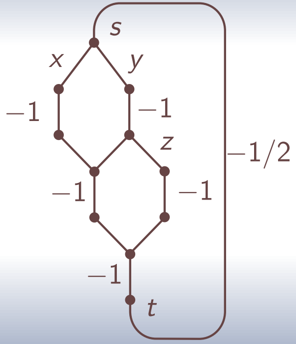
Graph $\implies$ Matrix

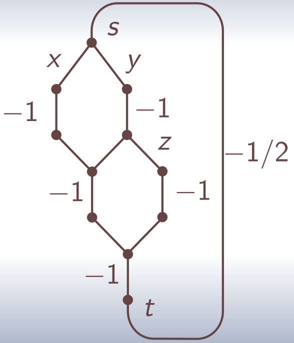


Determinant

$\mathfrak{S}_n$ = Permutation group of $\{1, \ldots, n\}$

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

Graph $\Longrightarrow$ Matrix



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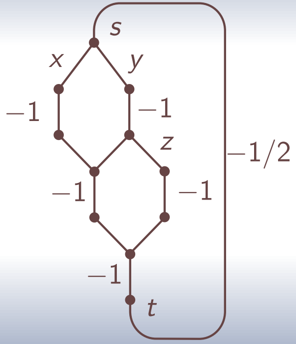

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- Up to signs, $\det A$ = sum of weights of cycle covers in G'

Summary

$P(x_1, \ldots, x_n)$

Weakly-Skew Circuit

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$$P(x_1, \ldots, x_n) = \sum_{s\text{–}t \text{ path } P} (-1)^{\frac{|P|-1}{2}} w(P)$$

Weakly-Skew Circuit

Arithmetic Branching Program

Summary

$P(x_1,\dots,x_n)$	Weakly-Skew Circuit
$= \sum_{s\text{–}t \text{ path } P} (-1)^{\frac{\lvert P\rvert-1}{2}} w(P)$	Arithmetic Branching Program
$= \sum_{\text{cycle cover } C} (-1)^{\operatorname{sgn}(C)} w(C)$	Graph G'

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$= \det \mathrm{Adj}(G')$	Symmetric Matrix

Summary

$$P(x_1,\ldots,x_n)$$ Weakly-Skew Circuit

$$= \sum_{s\text{–}t \text{ path } P} (-1)^{\frac{|P|-1}{2}} w(P)$$ Arithmetic Branching Program

$$= \sum_{\text{cycle cover } C} (-1)^{\mathrm{sgn}(C)} w(C)$$ Graph G'

$$= \det \mathrm{Adj}(G')$$ Symmetric Matrix

	Formula	Weakly-skew circuit
Non symmetric	$e+1$	$(e+i)+1$
Symmetric	$2e+1$	$2(e+i)+1$

Introduction

$$xy + yz + xz$$

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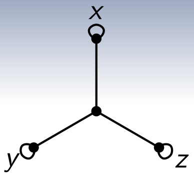

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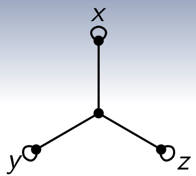

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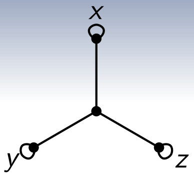

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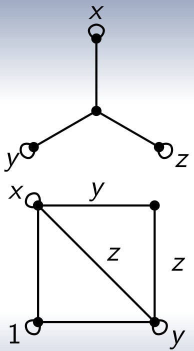

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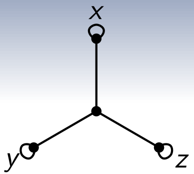

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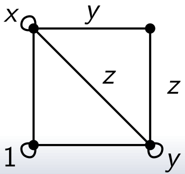

What about $xy + z$?

Representable polynomials

A polynomial is said **representable** if it has a SDR.

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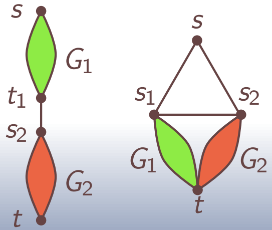

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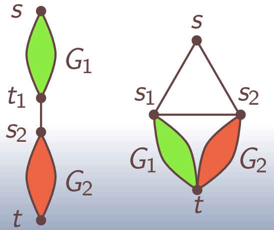
- $\det(G \setminus \{s,t\}) = 1$
- $\det(G \setminus \{s\}) = \det(G \setminus \{t\}) = 0$



A class of representable polynomials

Theorem

$L(x_1, \ldots, x_m) = P_0^2 + x_1 P_1^2 + \cdots + x_m P_m^2$ *is representable.*

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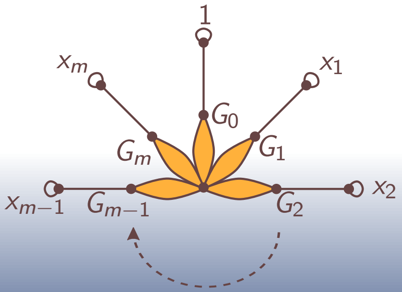

Obstructions to representability

Theorem

If P is representable, then

$$P \equiv L_1 \times \cdots \times L_k \bmod \langle x_1^2 + 1, \ldots, x_m^2 + 1 \rangle$$

where the L_i's are linear.

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Theorem

If P is multilinear, this is an equivalence.

Proof idea

- *Modulo* $\langle x_1^2 + \ell_1, \ldots, x_m^2 + \ell_m \rangle$: no variable **outside the diagonal**

$$xz + y^2 = \det \begin{pmatrix} x & y \\ y & z \end{pmatrix}$$

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Wait! Is $xy + z$ representable?

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3. If $P \equiv L(P') \times P'_0 \bmod \langle x_i^2\rangle$
   - Then IsRepresentable(P'_0)
   - Else Return False

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Problem

Problem [Bürgisser 00]

Is the partial permanent VNP-complete in characteristic 2?

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- VP, VNP, VNP-complete ≡ P, NP, NP-complete for polynomials

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Theorem (Malod'11, Valiant'02 via Mengel'11)

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Thank you!