Représentations des polynômes, algorithmes et bornes inférieures

Bruno Grenet

sous la direction de Pascal Koiran et Natacha Portier

Jeudi 29 novembre 2012

Representations of polynomials, algorithms and lower bounds

Bruno Grenet

supervised by Pascal Koiran and Natacha Portier

Thursday, November 29, 2012

Representation of Univariate Polynomials

$$P(X) = X^{10} - 4\,X^{8} + 8\,X^{7} + 5\,X^{3} + 1$$

Representations

- Dense:

$$[1, 0, -4, 8, 0, 0, 0, 5, 0, 0, 1]$$

- Sparse:

$$\Big\{(10:1), (8:-4), (7:8), (3:5), (0:1)\Big\}$$

Representation of Multivariate Polynomials

$$P(X,Y,Z) = X^2Y^3Z^5 - 4\,X^3Y^3Z^2 + 8\,X^5Z^2 + 5\,XYZ + 1$$

Representations

- Dense:

$$[1, \ldots, -4, \ldots, 8, \ldots, 5, \ldots, 1]$$

- Lacunary (supersparse):

$$\Big\{(2,3,5:1), (3,3,2:-4), (5,0,2:8), (1,1,1:5), (0:1)\Big\}$$

Representation of Multivariate Polynomials

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- Lacunary (supersparse):

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Arithmetic Circuits

$$Q(X,Y,Z) = X^4 + 4X^3Y + 6X^2Y^2 + 4XY^3 + X^2Z + 2XYZ + Y^2Z + X^2 + Y^4 + 2XY + Y^2 + Z^2 + 2Z + 1$$

Arithmetic Circuits

$$Q(X,Y,Z) = (X+Y)^4 + (Z+1)^2 + (X+Y)^2(Z+1)$$

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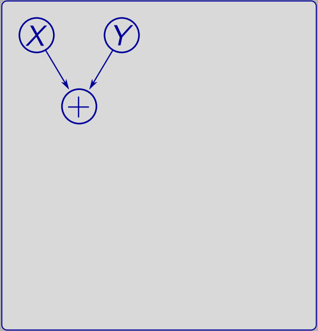

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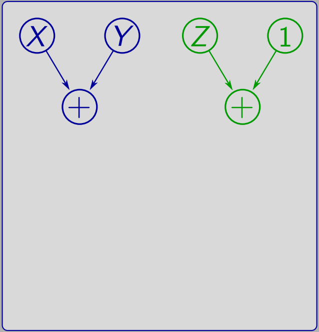

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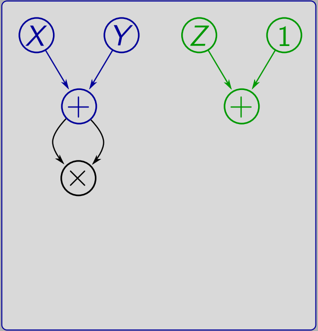

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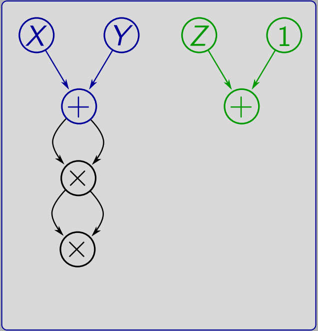

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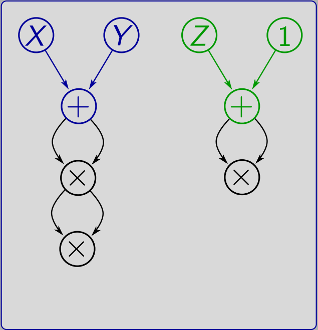

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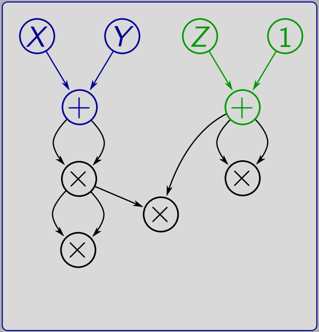

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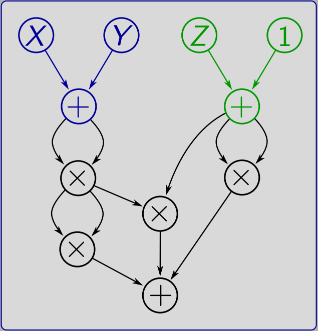
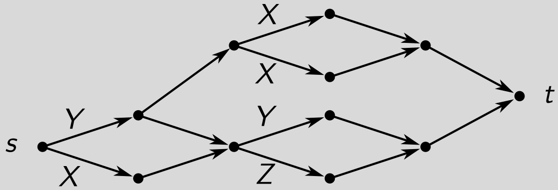


Arithmetic Branching Programs



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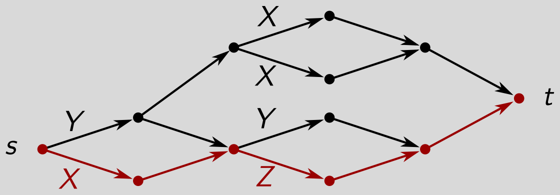
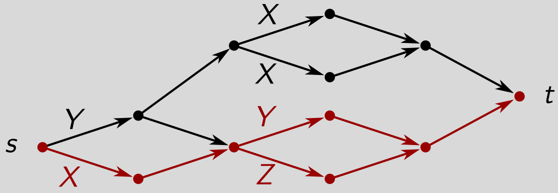


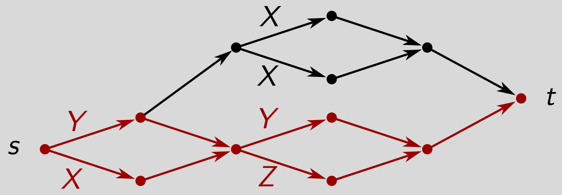
XZ

Arithmetic Branching Programs



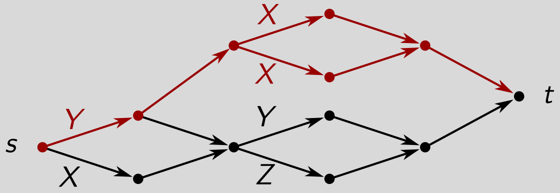
$$X(Y+Z)$$

Arithmetic Branching Programs



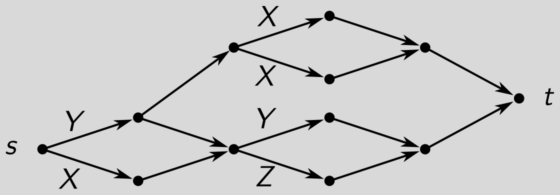
$$(X + Y)(Y + Z)$$

Arithmetic Branching Programs



$$2XY$$

Arithmetic Branching Programs



$$2XY + (X + Y)(Y + Z)$$

Some questions

- Links between representations

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  - Circuits
  - Branching programs
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  - Factorization — lacunary
  - Polynomial Identity Testing — circuit

Outline

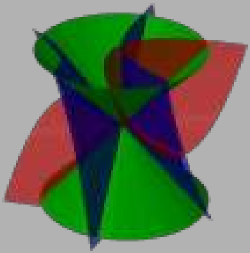
1. Resolution of polynomial systems

2. Determinantal Representations of Polynomials

3. Factorization of lacunary polynomials

1. Resolution of polynomial systems

Is there a (nonzero) solution?



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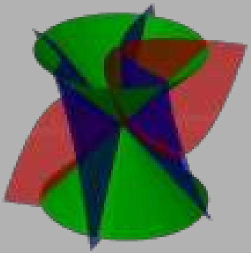

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Input: System of polynomials $f = (f_1, f_2, f_3)$, $f_j \in \mathbb{Z}[X, Y, Z]$, **homogeneous**

Question: Is there a point $a = (a_1, a_2, a_3) \in \mathbb{C}^3$, **nonzero**, s.t. $f_1(a) = f_2(a) = f_3(a) = 0$?

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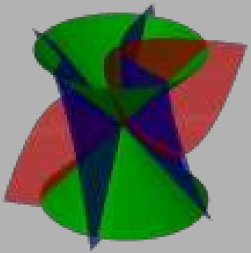

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More on the homogeneous case

Input: $f_1, \dots, f_s \in \mathbb{K}[X_0, \dots, X_n]$, homogeneous

Question: Is there a nonzero $a \in \bar{\mathbb{K}}^{n+1}$ s.t. $f(a) = 0$?

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  $\rightsquigarrow$ Trivial? Easy? Hard?

Definitions

PolSys($\mathbb{K}$)

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Resultant($\mathbb{K}$)

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Upper bounds

Proposition (Koiran'96)

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Corollary

Under GRH, $\textsc{HomPolSys}(\mathbb{Z})$ and $\textsc{Resultant}(\mathbb{Z})$ belong to AM.

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Proof. Remove the unwanted zero root: Add $\sum_i X_i Y_i - 1$ to the system. $\square$

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Positive characteristics

If p is prime, $(\textsc{Hom})\textsc{PolSys}(\mathbb{F}_p)$ & $\textsc{Resultant}(\mathbb{F}_p)$ are in PSPACE.

Known lower bounds

Notation: $\mathbb{F}_0 = \mathbb{Q}$

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For $p = 0$ or prime, $\textsc{PolSys}(\mathbb{F}_p)$ & $\textsc{HomPolSys}(\mathbb{F}_p)$ are **NP-hard**.

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For $p = 0$ or prime, PolSys($\mathbb{F}_p$) & HomPolSys($\mathbb{F}_p$) are NP-hard.

Proposition (Folklore, see Heintz-Morgenstern'93)

Resultant($\mathbb{Z}$) is NP-hard.

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$\mathbb{F}_p$	NP-hard	NP-hard	**Open**

- What happens for Resultant($\mathbb{F}_p$), $p > 0$?

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Theorem (G.-Koiran-Portier'10-12)

Let p be a prime number.

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Theorem (G.-Koiran-Portier'10-12)

Let p be a prime number.

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- Resultant($\mathbb{F}_q$) is NP-hard for **dense** polynomials for some $q = p^s$.

Proof idea

$f(X)$: s degree-2 homogeneous polynomials in $\mathbb{F}_p[X_0, \ldots, X_n]$

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From $f(X)$ to $g(X, Y)$

$$g(X,Y) = \begin{pmatrix} f_1(X) & \\ \vdots & \text{(unchanged)} \\ f_n(X) & \\ & \\ & \\ & \end{pmatrix}$$

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$f(X)$: s degree-2 homogeneous polynomials in $\mathbb{F}_p[X_0, \dots, X_n]$

From $f(X)$ to $g(X,Y)$

$$g(X,Y) = \begin{pmatrix} f_1(X) \\ \vdots \\ f_n(X) \\ f_{n+1}(X) + \lambda Y_1^2 \\ f_{n+2}(X) - Y_1^2 + \lambda Y_2^2 \\ \vdots \\ f_{s-1}(X) - Y_{s-n-2}^2 + \lambda Y_{s-n-1}^2 \\ f_s(X) - Y_{s-n-1}^2 \end{pmatrix}$$

- $f(a) = 0 \implies g(a,0) = 0$

Proof idea

$f(X)$: s degree-2 homogeneous polynomials in $\mathbb{F}_p[X_0,\dots,X_n]$

From $f(X)$ to $g(X,Y)$

$$g(X,Y) = \begin{pmatrix} f_1(X) & & \\ \vdots & & \\ f_n(X) & & \\ f_{n+1}(X) & & +\lambda Y_1^2 \\ f_{n+2}(X) & - Y_1^2 & +\lambda Y_2^2 \\ \vdots & & \\ f_{s-1}(X) & - Y_{s-n-2}^2 & +\lambda Y_{s-n-1}^2 \\ f_s(X) & - Y_{s-n-1}^2 & \end{pmatrix}$$

- $f(a) = 0 \implies g(a,0) = 0$
- Find λ such that $(g(a,b) = 0 \implies b = 0)$

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- $f(a) = 0 \implies g(a, 0) = 0$
- Find λ such that $(g(a, b) = 0 \implies b = 0 \implies f(a) = 0)$

Conclusion

- **NP-hardness results** for square homogeneous systems of polynomials over finite fields

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Main open problem

- Improve the PSPACE upper bound in positive characteristics…
- … or the NP lower bound.

2. Determinantal Representations of Polynomials

Determinant

Definition

$\mathfrak{S}_n$ = permutations of $\{1, \ldots, n\}$

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

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$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

- Complexity of the determinant

Determinantal representations

$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

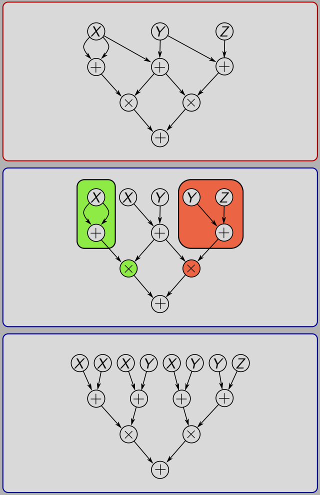
- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "P = NP?"

Determinantal representations

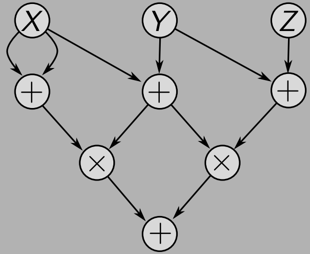
$$2XY+(X+Y)(Y+Z) = \det \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & Y & 0 & 1 & 0 & 1 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{vmatrix}$$

- Complexity of the determinant
- Determinant vs. Permanent: Algebraic "$P = NP$?"
- Links between circuits, ABPs and the determinant

Circuits



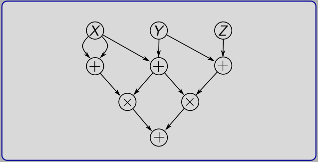
$$2X(X+Y)+(X+Y)(Y+Z)$$

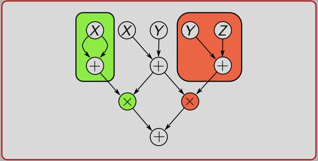


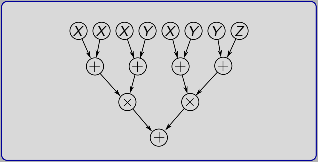
Arithmetic circuit

Size	6
Inputs	3

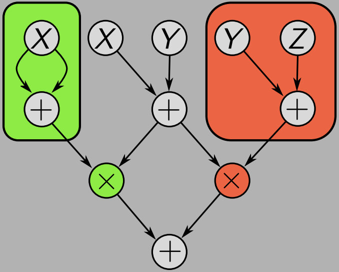
Circuits







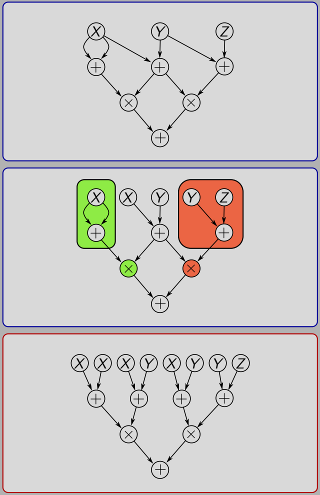
$$2X(X+Y)+(X+Y)(Y+Z)$$



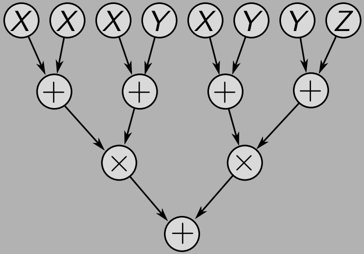
Weakly-skew circuit

Size	6
Inputs	5

Circuits



$$2X(X+Y)+(X+Y)(Y+Z)$$



Formula

Size	7
Inputs	8

Results

Proposition (Valiant'79)

Formula of **size** $s \rightsquigarrow$ Determinant of a matrix of **dimension** $(s+2)$

Results

Proposition (Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11)

Formula of size $s \rightsquigarrow$ Determinant of a matrix of dimension $(s+1)$

Results

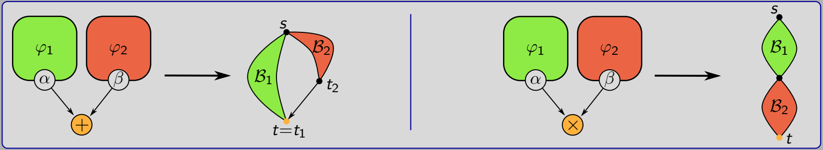
Proposition (Liu-Regan'06, G.-Kaltofen-Koiran-Portier'11)

Formula of **size** $s \rightsquigarrow$ Determinant of a matrix of **dimension** $(s+1)$

Proposition (Toda'92, Malod-Portier'08)

Weakly-skew circuit of **size** s with i **inputs**
$\rightsquigarrow$ Determinant of a matrix of **dimension** $(s+i+1)$

From Formulas to Branching Programs



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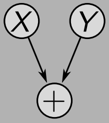
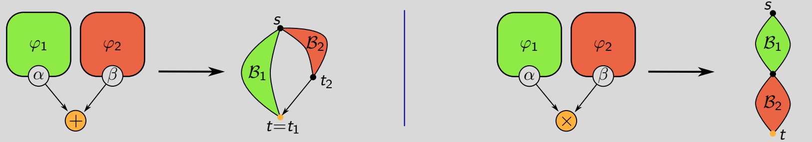
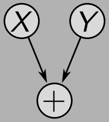
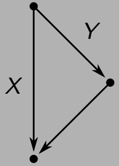
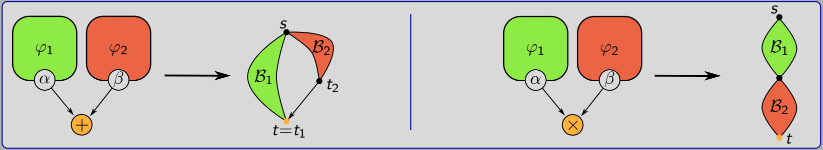




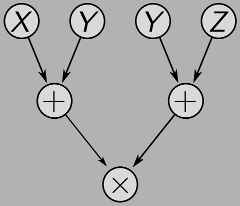
From Formulas to Branching Programs

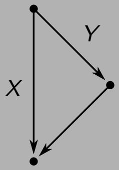


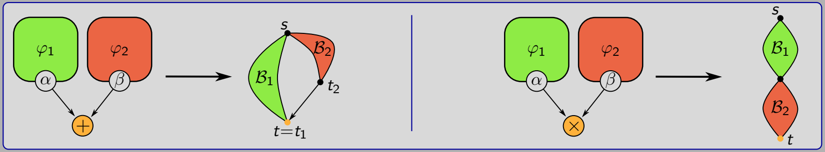




From Formulas to Branching Programs







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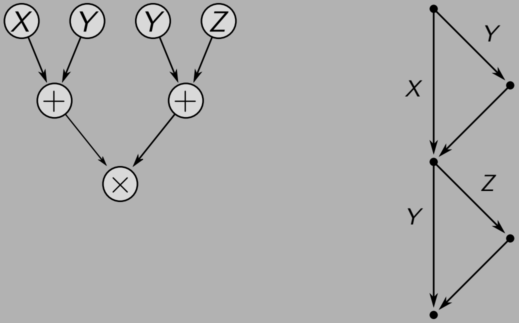
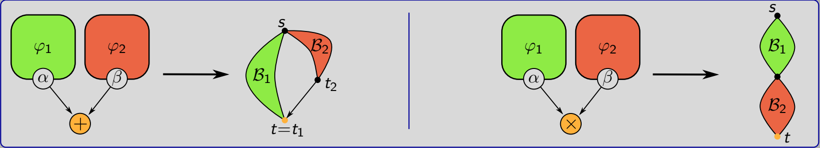
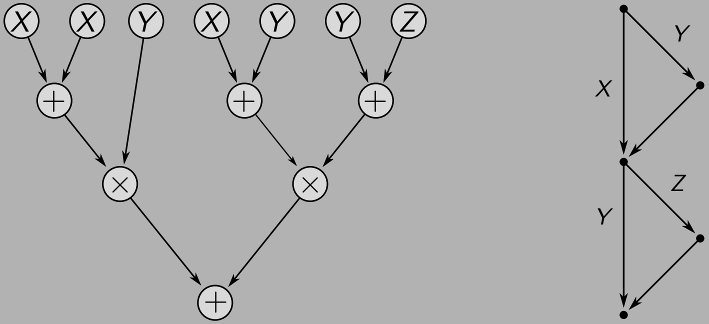
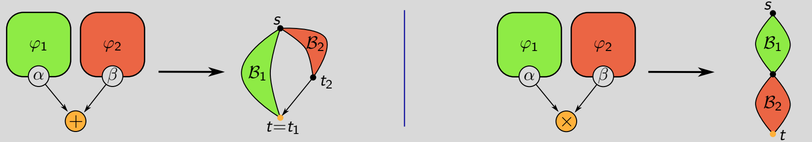




From Formulas to Branching Programs





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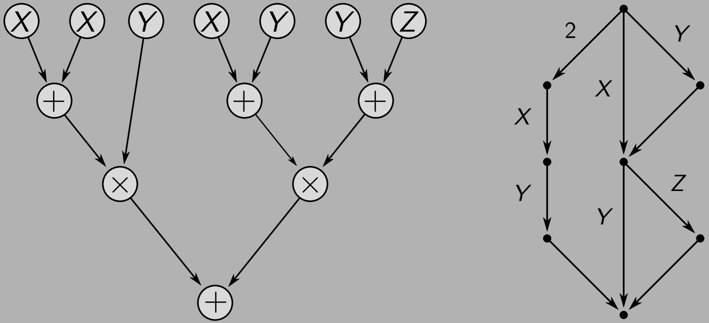

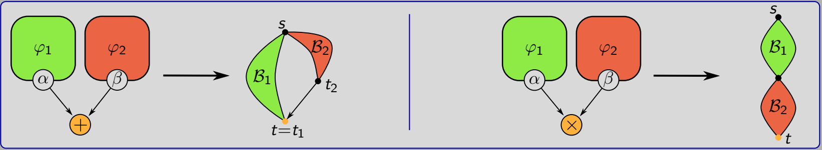

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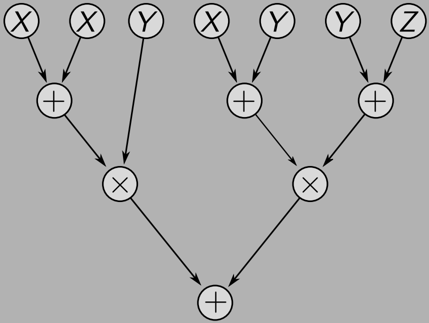
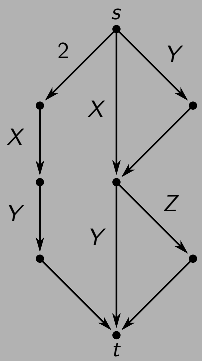
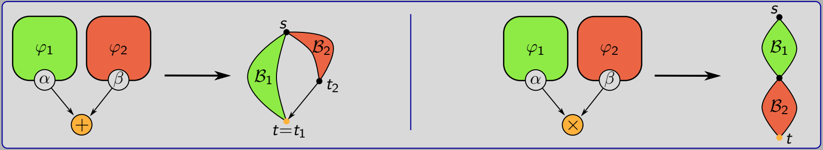
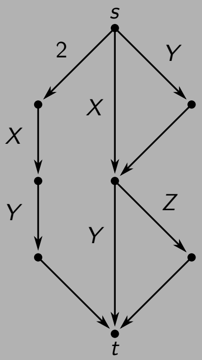


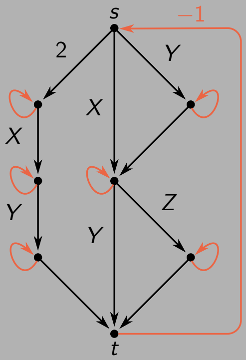




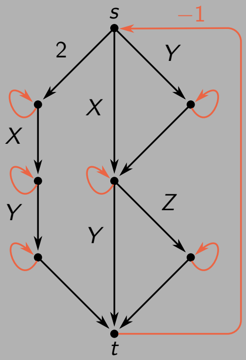
From Branching Programs to Determinants



From Branching Programs to Determinants

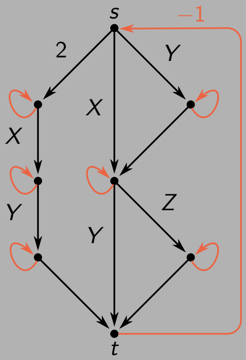


From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

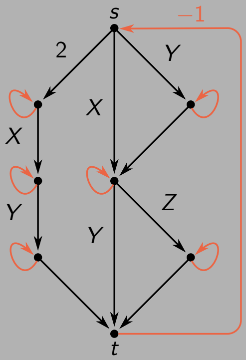
From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det M = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} M_{i,\sigma(i)}$$

From Branching Programs to Determinants

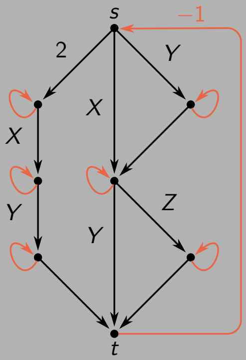


$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det M = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} M_{i,\sigma(i)}$$

- **Cycle covers** $\iff$ **Permutations**

From Branching Programs to Determinants



$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & Y & X & 0 & 0 \\ 0 & -1 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & Z & Y \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det M = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} M_{i,\sigma(i)}$$

- **Cycle covers** $\iff$ **Permutations**
- Up to signs, $\det(M) =$ **sum of the weights** of the cycle covers of G

Branching Program for the Permanent

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Branching Program for the Permanent

$$\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

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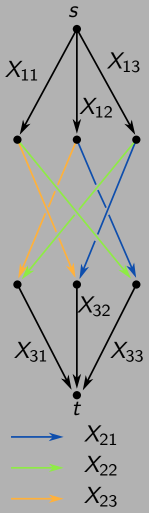

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Formula of **size** $s \rightsquigarrow$ Determinant of a matrix of **dimension** $(s+1)$

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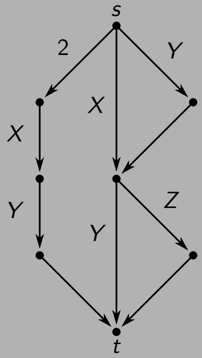
From Branching Programs to Symmetric Determinants



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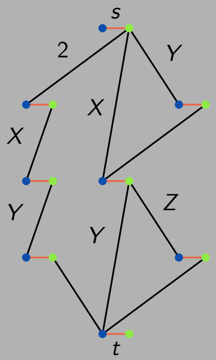

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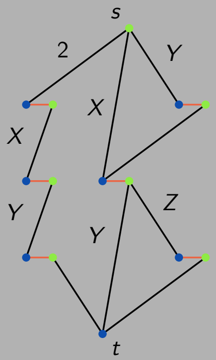

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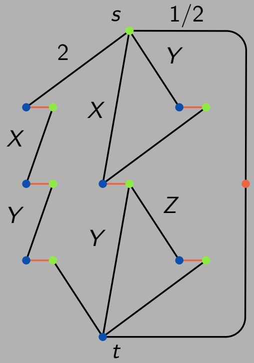

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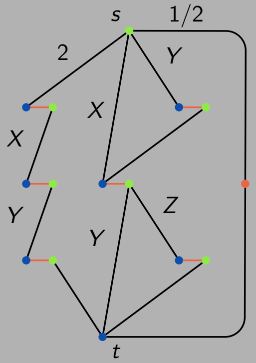

$$S = \begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & 0 & 0 & 0 & 0 & \frac{1}{2} \\
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & Z & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & -1 & 0 & 0 \\
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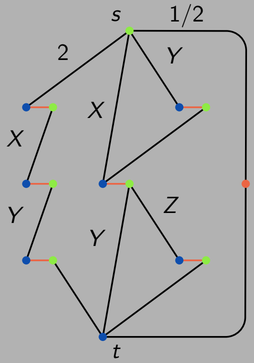

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\end{vmatrix}$$

Corollary

Let M be an $(n \times n)$ matrix. Then there exists a **symmetric matrix** S of **dimension** $\frac{2}{3}n^3 + o(n^3)$ s.t. $\det M = \det S$.

Conclusion

Same **expressiveness**:

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Main open question (Algebraic "P = NP?")

What is the **smallest** N s.t. the **permanent of dimension** n is a projection of the **determinant of dimension** N?

3. Factorization of lacunary polynomials

Introduction

$$-X^6 - X^2Y + X^5Y + XY^2 - X^4YZ - Y^2Z + X^4Z^2 + YZ^2$$

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$$\implies \text{size}(P) = \sum_{j=1}^{k} \text{size}(a_j) + \log(\alpha_{1j}) + \cdots + \log(\alpha_{nj})$$

Factorization of sparse univariate polynomials

$$P(X) = \sum_{j=1}^{k} a_j X^{\alpha_j} \qquad \mathrm{size}(P) = \sum_{j=1}^{k} \mathrm{size}(a_j) + \log(\alpha_j)$$

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Proposition (Lenstra'99)

Polynomial-time algorithm to find **factors of degree $\leq d$** if $a_j \in \mathbb{K}$, where $\mathbb{K}$ is an algebraic number field.

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Common ideas

Gap Theorem

$$P = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j} Y^{\beta_j}}_{P_0} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j} Y^{\beta_j}}_{P_1}$$

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$\mathrm{gap}(P)$: function of the **algebraic height** of P.

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Polynomial time algorithm to find **multilinear** factors of **bivariate** lacunary polynomials over algebraic number fields.

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Let $f_1, \dots, f_k \in \mathbb{K}[X]$. Then

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Proposition (Bôcher, 1900)

$\mathrm{W}(f_1, \dots, f_k) \neq 0 \iff$ the f_j's are linearly independent.

Wronskian & valuation

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Proof of the theorem.

$$\sum_{j=1}^{k} \alpha_j \geq \mathrm{val}(\mathrm{W}(f_1,\dots,f_k)) \geq \mathrm{val}(P) + \sum_{j=2}^{k} \alpha_j - \binom{k}{2}$$

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then $P \equiv 0$ iff both $P_0 \equiv 0$ and $P_1 \equiv 0$.

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with $u, v \neq 0$, $\alpha_1 \leq \cdots \leq \alpha_k$. If ℓ is the smallest index s.t.

$$\alpha_{\ell+1} > \alpha_1 + \binom{\ell}{2},$$

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Gap theorem

$$P(X, uX+v) \equiv 0$$
$$\iff P_1(X, uX+v) \equiv \cdots \equiv P_s(X, uX+v) \equiv 0$$

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$(Y - uX - v)$ divides $P(X, Y) \iff P(X, uX + v) \equiv 0$

- PIT algorithm $\rightsquigarrow$ test a **given** linear factor
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Main open problem

Extend to low-degree factors of multivariate polynomials

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