Lecture 6. Key exchange

Passive adversaries, agreeing on a secret

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https://membres-ljk.imag.fr/Bruno.Grenet/IntroCrypto.html

Introduction to cryptology
Université Grenoble Alpes – IM²AG
M1 INFO, MOSIG & AM

Introduction

Up to now: Symmetric cryptography

- Symmetric encryption *confidentiality*
- Message authentication codes *authenticity/integrity*

$\rightarrow$ Both require parties to share a common secret

How do two parties agree on a common secret?

Solutions

- Each pair of parties meets once (in person!) to agree on a key
  - Not always possible to meet
  - Each party retain a key for each other party N parties $\rightarrow \Theta(N^2)$ keys
- Key distribution centers:
  - Each party shares a (secret) key with the KDC, responsible to *distribute* session keys
  - Pros: Each party needs to meet only the KDC once and retain one key
  - Cons: the KDC is the central security point, and parties must meet the KDC

Public-key cryptography

Key-exchange protocols

- Two parties discuss publicly
- At the end, both parties know a same secret k
- External observers do not learn the secret, even after reading all exchanged messages

Public-key encryption and signatures

- Direct protocols to ensure confidentiality, authenticity and/or integrity
- Based on a pair (public key, private key) $\rightarrow$ no common secret

In this course

- Lecture 6: Key-exchange protocols
- Lecture 7: Public-key encryption
- Lecture 8: Signatures *public-key equivalent to MACs*

Contents

The goal of a key exchange

Allow two parties to agree on a key, remotely

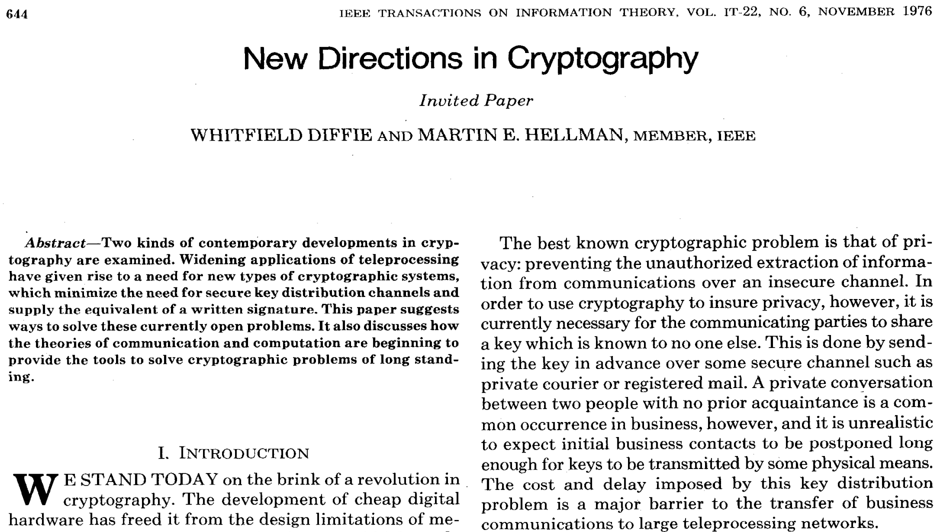
Objective

- Alice and Bob *publicly* exchange messages
- At the end of the exchange, they both know the same key k
- An attacker who sees all the messages has no information about k

Is this possible?

- The attacker sees as much as Alice and Bob ‽
- No information $\rightarrow$ computational security

New Directions in Cryptography



644 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. IT-22, NO. 6, NOVEMBER 1976

New Directions in Cryptography

Invited Paper

WHITFIELD DIFFIE AND MARTIN E. HELLMAN, MEMBER, IEEE

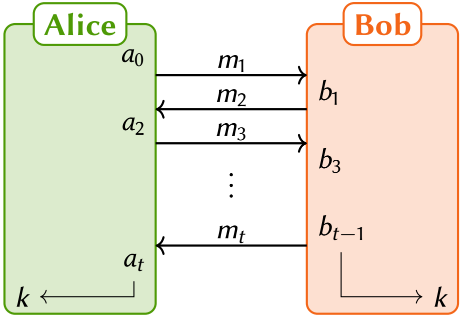
***Abstract*—Two kinds of contemporary developments in cryptography are examined. Widening applications of teleprocessing have given rise to a need for new types of cryptographic systems, which minimize the need for secure key distribution channels and supply the equivalent of a written signature. This paper suggests ways to solve these currently open problems. It also discusses how the theories of communication and computation are beginning to provide the tools to solve cryptographic problems of long standing.**

I. Introduction

WE STAND TODAY on the brink of a revolution in cryptography. The development of cheap digital hardware has freed it from the design limitations of me-

The best known cryptographic problem is that of privacy: preventing the unauthorized extraction of information from communications over an insecure channel. In order to use cryptography to insure privacy, however, it is currently necessary for the communicating parties to share a key which is known to no one else. This is done by sending the key in advance over some secure channel such as private courier or registered mail. A private conversation between two people with no prior acquaintance is a common occurrence in business, however, and it is unrealistic to expect initial business contacts to be postponed long enough for keys to be transmitted by some physical means. The cost and delay imposed by this key distribution problem is a major barrier to the transfer of business communications to large teleprocessing networks.

Definition of a protocol



Key exchange protocol

- Public : messages $m_1, \ldots, m_t$; key space $\mathcal{K}$
- Private : the a_is known only to Alice, the b_is only to Bob
- The protocol is correct if Alice and Bob compute the same key $k \in \mathcal{K}$

Vocabulary

- $m_1, \ldots, m_t$: the *transcript*

Security of a protocol

Secure key exchange protocol: given $m_1, \dots, m_t$, it is difficult to compute k

Game: key exchange indistinguishability

Challenger	simulates the protocol $\rightsquigarrow$ transcript $m_1, \dots, m_t$ and key $k \in \mathcal{K}$
(exp. b)	returns $\hat{k} = k$ if $b = 1$ and $\hat{k} \twoheadleftarrow \mathcal{K}$ if $b = 0$
Adversary	sees the transcript and $\hat{k}$, and returns a bit b'

Advantages

- For a specific adversary $\mathcal{A}$:

$$\mathrm{Adv}_{\mathrm{KE}}^{\mathrm{IND-EAV}}(\mathcal{A}) = \left|\Pr\left[b' = 1 \middle| b = 1\right] - \Pr\left[b' = 1 \middle| b = 0\right]\right|$$

- For the protocol:

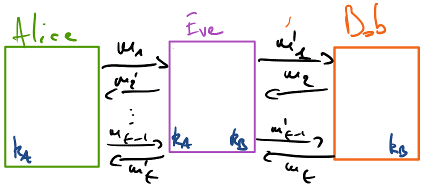
$$\mathrm{Adv}_{\mathrm{KE}}^{\mathrm{IND-EAV}}(t) = \max_{\mathcal{A}_t} \mathrm{Adv}_{\mathrm{KE}}^{\mathrm{IND-EAV}}(\mathcal{A}_t)$$

where $\mathcal{A}_t$ denotes an algorithm with running time $\leq t$

Eavesdropper security and *person-in-the-middle* attack

Indistinguishability in the presence of an eavesdropper

- Security definition assumes an authenticated channel between Alice and Bob
- The adversary is only *passive*



Person-in-the-middle attack

- Eve intercepts messages between Alice and Bob
- She impersonates both Alice and Bob
- She creates a common secret with Alice, and another one with Bob
- Alice and Bob incorrectly think they share a common secret

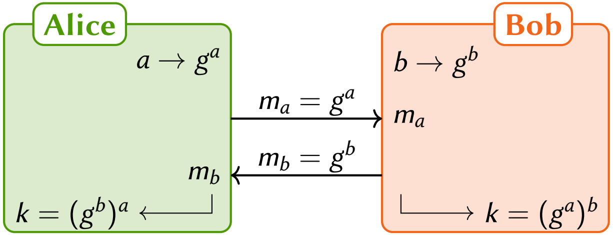
Key exchange is not enough

Combine with authentication

signatures

- *Authenticated* key exchange

A glimpse of Diffie-Hellman protocol



Public:	a *number* g
Alice	chooses a *random* a
	computes g^a
	sends $m_a = g^a$ to Bob
Bob	chooses a *random* b
	computes g^b
	sends $m_b = g^b$ to Alice
Alice	computes $k = m_b^a = g^{ab}$
Bob	computes $k = m_a^b = g^{ab}$

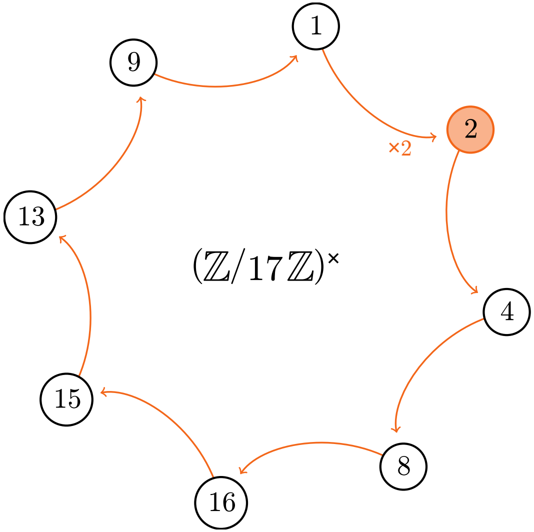
Correction: $k_A = (g^b)^a = g^{a \cdot b} = (g^a)^b = k_B$

Remaining questions

- What is this *number* g?
- How can this scheme be secure while Eve sees g^a and g^b?

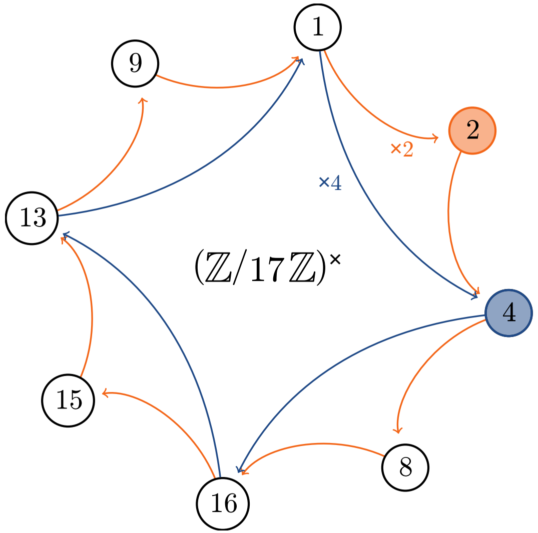
Contents

The multiplicative group of $\mathbb{Z}/p\mathbb{Z}$



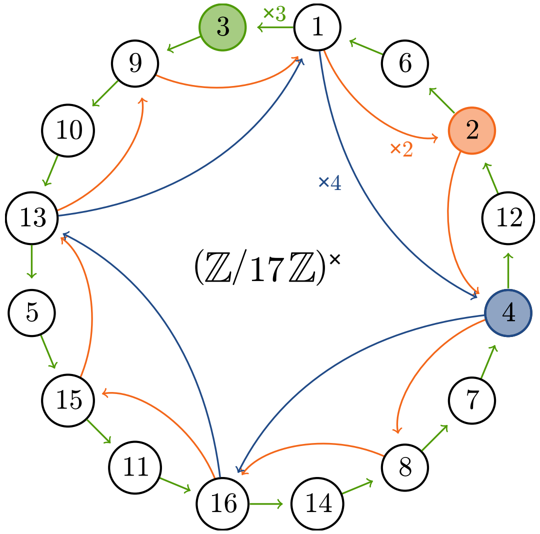
- 2 has order 8

The multiplicative group of $\mathbb{Z}/p\mathbb{Z}$



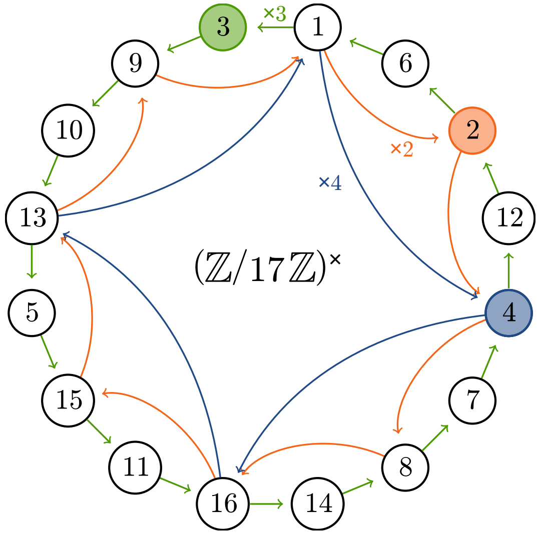
- 2 has order 8
- 4 has order 4

The multiplicative group of $\mathbb{Z}/p\mathbb{Z}$



- 2 has order 8
- 4 has order 4
- 3 has order 16 $\longrightarrow$ generator

The multiplicative group of $\mathbb{Z}/p\mathbb{Z}$



- 2 has order 8
- 4 has order 4
- 3 has order 16 $\longrightarrow$ generator

Theorem
For every p, $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group: there exists a generator $g \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that

$$(\mathbb{Z}/p\mathbb{Z})^\times = \{g^n : 0 \leq n < p-1\}$$

Remarks
- The generator is *not* unique (ex.: 3, 5, 6, 7, 10, 11, 12, 14)
- $g^n = g^{n \bmod p-1}$ since $g^{p-1} = 1$
  $(\mathbb{Z}/p\mathbb{Z})^\times = \{g^n : n \in \mathbb{Z}\}$

Cyclic groups

Definition

A (multiplicative) cyclic group of order n with generator g is a set $\langle g\rangle = \{g^i : 0 \leq i < n\}$ of size n with $g^n = g^0$ and *standard* rules for exponents: $g^{a+b} = g^a \cdot g^b$, $(g^a)^b = g^{ab}, \ldots$

Remarks

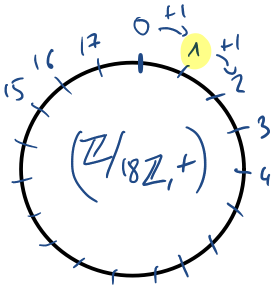
- The generator is not unique
- For all $m \in \mathbb{Z}$, $g^m = g^{m \bmod n}$ $\qquad \langle g\rangle = \{g^t : t \in \mathbb{Z}\}$
- Each element $x \in \langle g\rangle$ *generates a subgroup* $\langle x\rangle = \{x^t : t \in \mathbb{Z}\} \subset \langle g\rangle$
  - order of x = order of $\langle x\rangle$ = number of elements in $\langle x\rangle$

More general definitions

- Additive cyclic group with generator g: $G = \{i \cdot g : 0 \leq i < n\}$
  - Other binary operations
- Non-cyclic groups $\qquad$ Lecture 9
- Infinite (cyclic or non-cyclic) groups

Examples – graphical representations

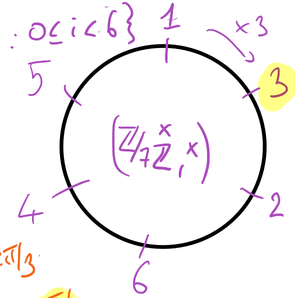
Additive

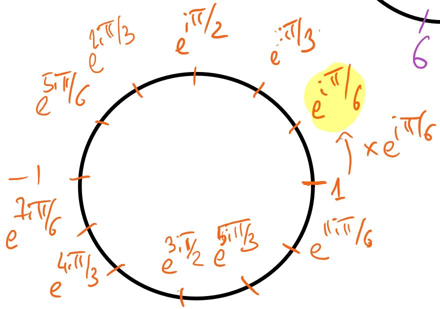


$$(\mathbb{Z}/18\mathbb{Z},+) = \{ i \times 1 : 0 \le i < 18 \}$$

Multiplicative

$$\mathbb{Z}/7\mathbb{Z}^{\times} = \{ 3^i : 0 \le i < 6 \}$$





Discrete logarithm problem

Definitions

Given a cyclic group $G = \langle g \rangle$,

- the discrete logarithm of x in base g is the unique $0 \leq t < \#G$ such that $x = g^t$
- the discrete logarithm problem is: given x, compute t

The naive algorithm

- Compute $g^0, g^1, g^2, \ldots$ until we get $g^t = x$
- Complexity $O(t) = O(\#G)$ operations in G

Easy case: $(\mathbb{Z}/n\mathbb{Z}, +)$

- Generators: 1 or any g such that $\operatorname{GCD}(g, n) = 1$
- Discrete logarithm of x: t s.t. $x = t \cdot g \bmod n$
- Case $g = 1$: nothing to do!
- General case:
  1. Compute u, v s.t. $u \cdot g + v \cdot n = 1$ — Extended Euclidean Algorithm
  2. Return $t = u \cdot x \bmod n$ — $t \cdot g = u \cdot x \cdot g = x \bmod n$

Baby step-giant step: A picture is worth a thousand words

(Shanks, 1971)

Inputs: g (generator of G)

$x \in G$

Output: t s.t. $x = g^t$

$G = \mathbb{Z}/_{17}\mathbb{Z}^{\times}$, $g = 3$, $x = 7$

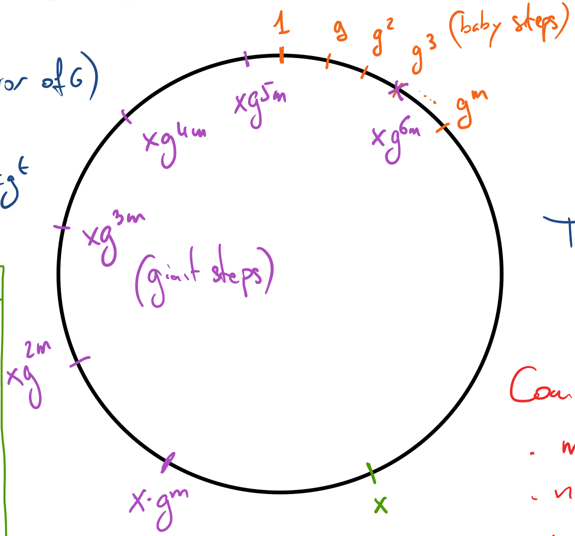
$(m=4)$

BS: $[1, 3, 9, 10, 13]$

GS: $[7, 6, 10]$

$3^3 = x \cdot 3^8$

$\Rightarrow x = 3^{-5} = 3^{11}$



I have found that

$$g^3 = x g^{6m}$$

Therefore

$$x = g^{\overset{t}{(3-6m) \bmod (\#G)}}$$

Complexity:

- m multiplications for the baby steps
- n/m — giant steps

↳ If $m \simeq \sqrt{n}$: $\mathcal{O}(\sqrt{n})$ multiplications

Baby step-giant step: the algorithm

(Shanks, 1971)

Input: a cyclic group $G = \langle g \rangle$ of order n and $x \in G$

Output: the discrete logarithm t of x in base g

```
1. $m \leftarrow \lceil \sqrt{n} \rceil$
2. $B \leftarrow [1, g, g^2, \ldots, g^{m-1}]$                    *Baby steps*
3. $(h, y, j) \leftarrow (g^m, x, 0)$
4. while $y \notin B$: $(y, j) \leftarrow (y \cdot h, j + 1)$       *Giant steps*: $y = x \cdot g^{m \cdot j}$
5. $i \leftarrow$ index such that $y = g^i$                        *Match found*: $x \cdot g^{m \cdot j} = g^i$
6. return $(i - m \cdot j) \bmod n$
```

Analysis

Correction: by Euclidean division, there exist $i, j < m$ such that $t = i - mj \bmod n$

Complexity: $O(\sqrt{n})$ baby steps + $O(\sqrt{n})$ giant steps

Test for a match at each (giant) step: $O(1)$ using hash tables

$\Rightarrow O(\sqrt{n})$ (same in space)

DLP hardness

Theorem (baby-step giant-step)

In any cyclic group G, the discrete logarithm problem can be computed in time $O(\sqrt{\#G})$

- Easily parallelizable
- Variants with better space complexity: Pollard's ρ or kangaroos (*a.k.a.* λ) algorithms
- Pohlig-Hellman (1978): $O(\sqrt{p})$ *largest prime divisor of* $\#G$

Choice of G

- $(\mathbb{Z}/n\mathbb{Z}, +)$ or $\#G$ small: easy DLP!
- $(\mathbb{Z}/p\mathbb{Z}^\times, \times)$: usually hard, though not *maximally* hard $\ll \sqrt{p}$
  - record: p of 795 bits, 3100 core-year Boudot *et al.*, 2019
- Points of an elliptic curve over a finite field: maximally hard $O(\sqrt{n})$
  - record: group of 114 bits, 13 days on GPU Zieniewicz & Pons, 2020

Additional remarks

- One should use prime order groups
- Algorithm polynomial in $\log n$ on a quantum computer Shor, 1997

Final words on the discrete logarithm

Example of a (conjecturally) *one-way* function

Informal definition

A function $f : E \to F$ is *one-way* if

- it is efficiently computable
- its inverse is hard to compute

given $y \in F$, find x s.t. $f(x) = y$

Exponentiation in a cyclic group

$\textsc{pow} : \mathbb{Z} \to \langle g \rangle$ defined by $\textsc{pow}(n) = g^n$

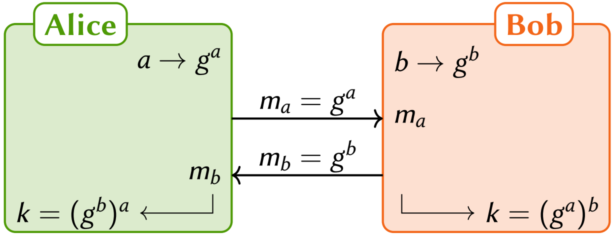
- Efficiently computable using *binary powering*
  - $\textsc{pow}(n) = \begin{cases} \textsc{pow}(\lfloor n/2 \rfloor)^2 & \text{if } n \text{ is even} \\ g \cdot \textsc{pow}(\lfloor n/2 \rfloor)^2 & \text{otherwise} \end{cases}$
- Inverse of $\textsc{pow}$ is the DLP

$O(\log n)$ operations in $\langle g \rangle$

$\Theta(\sqrt{n})$

Contents

The protocol



Diffie-Hellman protocol

Input :	Group $G = \langle g \rangle$ of order n
Alice	draws $a \twoheadleftarrow \{0, \ldots, n-1\}$, computes $m_a = g^a$ and sends m_a to Bob
Bob	draws $b \twoheadleftarrow \{0, \ldots, n-1\}$, computes $m_b = g^b$ and sends m_b to Alice
Alice	computes $k_a = m_b^a$
Bob	computes $k_b = m_a^b$

Correctness

The protocol is correct: $k_a = (g^b)^a = g^{ab} = (g^a)^b = k_b$

Use in practice

Where do the secret lives?

- Shared secret $k \in G$, while usually one needs it in $\{0,1\}^*$
- *Key derivation function* KDF : $G \to \{0,1\}^*$ $\simeq$ hash function

Person-in-the-middle attack *a.k.a Man-in-the-middle*

- Requires an authentication between Alice and Bob
- Out-of-scope of this lecture $\to$ *c.f.* signatures Lecture 8

Cost of the protocol

- Requires two exponentiations in G
  - $O(\log \#G)$ operations in G *binary powering*
  - For instance: $O(\log^2 p \log\log p)$ bit-operations for $\mathbb{Z}/p\mathbb{Z}$

Three difficulty assumptions

G: fixed order-n group with known generator g

Discrete-Log-Problem Game

Challenger	samples $t \twoheadleftarrow \{0, \ldots, n-1\}$ and computes $h = g^t$
Adversary	is given h and tries to compute t

Computational Diffie-Hellman Game

Challenger	samples $a, b \twoheadleftarrow \{0, \ldots, n-1\}$ and computes $m_a = g^a$ and $m_b = g^b$
Adversary	is given m_a and m_b and tries to compute g^{ab}

Decisional Diffie-Hellman Game

Challenger	samples $a, b, c \twoheadleftarrow \{0, \ldots, n-1\}$, computes $m_a = g^a$, $m_b = g^b$
(exp. λ)	and $m_c = g^c$ if $\lambda = 0$, $m_c = g^{ab}$ if $\lambda = 1$
Adversary	is given m_a, m_b and m_c and tries to guess λ

DLP/CDH/DDH assumptions

- It is *hard* for an adversary with *reasonable* running-time to win the DLP game
- It is *hard* for an adversary with *reasonable* running-time to win the CDH game
- It is *hard* for an adversary with *reasonable* running-time to win the DDH game

Relating the three assumptions

Lemma

For any cyclic group $G = \langle g \rangle$, DDH is hard $\Rightarrow$ CDH is hard $\Rightarrow$ DLP is hard

Remark

Each game can be won by an adversary running in time $O(\sqrt{\#G})$

Proof of lemma by contrapositive

(1) Assume A_{DLP} is able to compute a discrete log.
Build A_{CDH}: given $m_a = g^a$, $m_b = g^b$, use A_{DLP} to compute a and b
and output $(g)^{ab}$.

(2) Assume A_{CDH} is able to compute g^{ab} from $m_a = g^a$ and $m_b = g^b$
Build A_{DDH}: given m_a, m_b and m_c, use A_{CDH} to compute g^{ab}
and compare $m_c \overset{?}{=} g^{ab}$

Security of the Diffie-Hellman protocol

Theorem

The Diffie-Hellman protocol in the group G is IND-EAV secure under the DDH assumption

Proof

The DDH game is <u>exactly</u> the same as the IND-EAV game for the DH protocol

Choice of group

DDH can be easier than DLP

- In $(\mathbb{Z}/p\mathbb{Z})^\times$, $g^{\frac{p-1}{2}} = -1$, therefore $(g^x)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd} \end{cases}$
- Adversary: Given $m_a = g^a$, $m_b = g^b$ and $m_c = g^c$, check if $m_a^{\frac{p-1}{2}} \cdot m_b^{\frac{p-1}{2}} = m_c^{\frac{p-1}{2}}$
  - $\Pr[a \cdot b \text{ even}] = \Pr[a \text{ even} \vee b \text{ even}] = \frac{3}{4}$
  - $\Pr[c \text{ even}] = \frac{1}{2}$ (when $c \leftarrow \{0, \ldots, p-2\}$)

Use prime-order (sub)groups

- Avoid the previous attack
- Complexity of DLP is anyway related to largest prime subgroup Pollig-Hellman

Examples

- Prime-order subgroup of $(\mathbb{Z}/p\mathbb{Z})^x$
- Points on elliptic curves
- …

Conclusion

50 shades of Diffie-Hellman

- The DH protocol is almost the only key exchange protocol
- But many choices of cyclic group G: $(\mathbb{Z}/p\mathbb{Z})^\times$, elliptic curve, isogenies, …
- Key derivation function to go from G to $\{0,1\}^*$

Security of the protocol

- Three hardness assumptions: DLP, CDH, DDH
  - DDH $\iff$ IND-EAV security
  - DDH $\Rightarrow$ CDH $\Rightarrow$ DLP
- Generic attack in $O(\sqrt{\#G})$ operations *baby steps – giant steps*

Inherent vulnerability: *person-in-the-middle*

- Charlie stands between Alice and Bob, and intercepts, modifies, etc. all messages between Alice and Bob
- Requires *authentication* between Alice and Bob
  - Use for instance signatures
  - *Authenticated Key Exchange*