Tutorial 09 – Bool Lyonnais(e)

Exercise 1.
For all the questions, give the size and the depth of the circuits you build.

1. Give a circuit computing the XOR of its two inputs.
2. Build a circuit computing $\textsc{sel}(x, y, z)$ defined by $\textsc{sel}(0, y, z) = y$ and $\textsc{sel}(1, y, z) = z$.
3. Let $f : \{0,1\}^n \to \{0,1\}^m$. We consider multiple-output circuits. (Equivalently, suppose we are given a circuit for each bit of output.) Show how to turn a multiple-output circuit computing f into a (classical) circuit deciding $\{(x, y) : y = f(x)\}$.
4. Build a multiple-output circuit computing the addition of two binary integers $x = \overline{x_{n-1} \cdots x_0}$ and $y = \overline{y_{n-1} \cdots y_0}$ using the schoolbook algorithm.
5. Suppose that n is a power of 2. Give a circuit computing $x + y$ using the following recursive algorithm: Compute $\overline{x_{n-1} \cdots x_{n/2}} + \overline{y_{n-1} \cdots y_{n/2}}$ and $\overline{x_{n-1} \cdots x_{n/2}} + \overline{y_{n-1} \cdots y_{n/2}} + 1$ in parallel, and use the carry-out of $\overline{x_{n/2-1} \cdots x_0} + \overline{y_{n/2-1} \cdots y_0}$ to choose the correct answer.
6. Compute the size and depth of the circuit we obtain for $\{(x, y, z) : x + y = z\}$. Propose another algorithm to decide this language. Is it better?

Exercise 2. *Shannon's Effect*

1. Show that every function from $\{0,1\}^n$ to $\{0,1\}$ can be represented by a CNF formula of size $n2^n$.
2. Show that a size-$O(2^n)$ circuit is sufficient.
3. (harder) Are you able to build a size-$O(2^n/n)$ circuit?
4. Show that there exist functions without size-$2^n/(10n)$ circuit.

Exercise 3. *Spira's Theorem*
A circuit is a *boolean formula* if every gate but the output in it has outdegree exactly 1. That is, the underlying graph is a tree.

✎ Show that for every size-t formula F, there exists an equivalent formula F' with depth at most $4 \log t$. What is the size fo F'? Can we obtain better bound if F' is a general circuit instead of a formula?

Exercise 4. *Monotony*
A boolean function $f : \{0, 1\}^n \mapsto \{0, 1\}$ is said monotonic if for every i and every $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$,

$$f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) \geq f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n).$$

✎ Show that a function is monotonic iff there exists a circuit with gates AND, OR, 0 and 1, but without NOT gate.