Partiel. Complexité algorithmique. 5 novembre 2010.
(*English version*)

Duration : 2h.
Documents (in particular lecture notes) are not allowed.

There are 6 independent exercises.
You can write in either French or English. Remember to justify your answers, this will be taken into account in the score.

Exercise 1.
Give the definition of the complexity classes P, NP and coNP, and of NP-complete and coNP complete languages. Give an example of a "natural" NP-complete language and of a "natural" coNP-complete language.

Exercise 2.
State the deterministic time hierarchy theorem and give a proof of it.

Exercise 3.
The notion of *non-deterministic polynomial time reducibility* is defined in the following way :
$L_1 \leq_{\mathsf{NP}} L_2$ iff there exists a non deterministic polynomial time Turing machine $\mathcal{M}$ with output tape , such that :
for any x in $\{0,1\}^*$, $x \in L_1$ iff there exists an execution of $\mathcal{M}$ on input x which produces an output y such that $y \in L_2$.
Show that the relation $\leq_{\mathsf{NP}}$ is transitive. Show that if $L_1 \leq_{\mathsf{NP}} L_2$ and $L_2 \in \mathsf{NP}$, then $L_1 \in \mathsf{NP}$.

Exercise 4.
We define the class P-Sel in the following way :
a langage $\mathcal{L}$ is in P-Sel (standing for selective P) if there exists $f : \{0;1\}^* \times \{0;1\}^* \to \{0;1\}^*$ computable in polynomial time and such that for any (x, y) :

$$\begin{cases} f(x,y) = x \text{ or } f(x,y) = y, \\ \text{if } x \in \mathcal{L} \text{ or } y \in \mathcal{L}, \text{ then } f(x,y) \in \mathcal{L} \end{cases}$$

We will then say that f is a *selection function* for $\mathcal{L}$.

1. Show that $\mathsf{P} \subseteq$ P-Sel.
2. Show that if $\mathcal{L} \in$ P-Sel, then so does its complement.
3. Prove that $\mathcal{L} \in$ P-Sel iff there exists L' in P such that : $\mathcal{L} \times \overline{\mathcal{L}} \subseteq L'$ and $\overline{\mathcal{L}} \times \mathcal{L} \subseteq \overline{L'}$, where $A \times B = \{\langle x, y \rangle \ : \ x \in A \text{ and } y \in B\}$, and $\overline{\mathcal{L}}$ stands for the complement of $\mathcal{L}$.
4. Show that if there exists a language in P-Sel which is NP-hard, then $\mathsf{P} = \mathsf{NP}$.

Exercise 5.

1. Let $c > 0$. Prove that $\mathsf{SPACE}(n^c) \subseteq \mathsf{NP}$ implies $\mathsf{NP} = \mathsf{PSPACE}$.
   Indication : one can use padding.
2. Deduce from that that for any $c > 0$, $\mathsf{SPACE}(n^c) \neq \mathsf{NP}$.
3. In the same way, show that $\mathsf{DTIME}(2^{cn}) \neq \mathsf{NP}$ for any $c > 0$.

Exercise 6.
We consider the following language :

$$IN\_PLACE\_DIVERGE = \{\alpha \in \{0,1\}^* \ / \ \text{on the empty input, the machine } \mathcal{M}_\alpha \text{ diverges and does not use more than } |\alpha| \text{ squares}\}.$$

where $\mathcal{M}_\alpha$ is the deterministic machine defined by the word α, and we say that a machine $\mathcal{M}$ *diverges on the input* x if its execution on this input does not terminate.

1. Show that this language belongs to PSPACE.
2. Show that it is PSPACE-complete.