Event-triggered Controllers using Contraction Analysis

Brigitte Bidegaray-Fesquet[1], Nacim Meslem[2], and Fairouz Zobiri[3]

[1] Univ. Grenoble Alpes, CNRS, Grenoble INP, LJK, 38000 Grenoble, France

[2] Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, 38000 Grenoble, France

[3] Department of Electrical Engineering (ESAT), KU Leuven, Leuven, 3001 Belgium

Abstract

In this paper, we introduce an event-triggered control method that relies on contraction analysis for Linear Time-Varying systems (LTV) and then extend it to a certain class of nonlinear systems. Contraction analysis considers stability with respect to a nominal trajectory rather than an equilibrium point. If two neighboring trajectories of a system are located in a contraction region, they will tend to each other and to a nominal trajectory. In the event-triggered control algorithm that we introduce, we suggest to update the control law whenever the system trajectory is about to leave the contraction region. We show that such a scheduling of the control law guarantees system stability, and we show that a minimum inter-event time exists between consecutive updates of the control law. We also show how to place the system trajectory in a contraction region and demonstrate that the classical controllability assumption on LTV systems is enough to ensure the existence of the required transformation to perform that.

1 Introduction

In most nowadays industrial applications, the information is collected from or sent to the plant via a network. Data travels through the network from the CPU to the plant, carrying the control signal, and from the sensors that measure plant output back to the CPU. In classical control theory, data transfer is carried out at a fixed rate. The transmission frequency is established by the Shannon–Nyquist theorem, and is generally set to a high value so as to remain close to the continuous shape of physical signals. If the CPU is in charge of a large number of plants, as is often the case, transmitting a large quantity of data at a very high rate can saturate the communications channels and induce packet losses with fatal consequences on the controlled plants.

Event-triggered control offers the possibility to reduce the communications between the CPU and the plant by sending a new value of the control law only if necessary. The procedure consists in first establishing a performance condition that the system has to respect, and the data collected by the sensors is used to compare the response of the system to the desired performance. If the response of the system falls within the range of acceptable performance, the control law is kept constant. If, on the contrary, the response of the systems violates the performance conditions, the CPU updates the control law and sends it to the plant. Several classes of systems have been addressed in the literature for both the continuous and discrete-time cases. For instance, in the continuous-time context, in [1, 4, 6, 12, 15, 16, 17, 19, 21, 22] the cases of linear and nonlinear systems are tackled and different approaches are proposed to design static and dynamic event-triggered mechanisms, while in [23, 24, 26] the case of switched systems is studied. Recently, in [13, 14] the case of linear time varying systems was considered, where the authors design event-triggered condition based on the theory of positive systems. On the other hand, in the discrete-time context, the case of linear systems is tackled in [2, 3] by using reachability analysis, while in [5] a discrete-time dynamic event-triggered policy is proposed for uncertain neural networks subject to time delays and disturbances. In [20] a novel error-to-actuator based event-triggered framework is proposed, where the tracking performance of the system can be preserved while reducing the number of signal transmissions.

The difficulty of event-triggered control resides in finding the appropriate performance measures, also called event-triggering conditions. By scheduling the control law according to these conditions, the system has to remain stable, while the communications between the controller and the plant have to be significantly reduced to see the benefits of this form of control. The task is even more complicated if the plant dynamics is nonlinear, in which case it is hard to find a set of performance measures that can be generalized to all systems as the structures of these systems can be disparate from one plant to another. Most of the methods found in the literature construct the event-triggering conditions on either the error between the current state and the state at the instant of the last event [19], or based on a Lyapunov function of the system [17, 21, 25], or both [16].

Both types of methods, however, present disadvantages. Methods based on the error produce a large number of events, as the error has to be kept small enough. Lyapunov methods help counter this problem but are difficult to generalize to nonlinear systems, for which, unlike for linear systems, no generic structure exists for Lyapunov functions. Therefore, these algorithms are hard to generalize to deal with nonlinear situations, and are challenging to parameterize.

In this work, instead of using the classical Lyapunov theory, we propose to apply a contraction analysis approach [8, 9, 10]. More precisely, unlike the classical methods where Lyapunov theory is applied to design event-triggering conditions and to analyze the system stability, in the framework of contraction theory, Lyapunov functions are used to characterize the contraction region of the system in new coordinates of the state vector. Under the controllability assumption, we first, consider the stabilization of linear time varying systems. A feature of contraction theory allows to transform a closed loop Linear Time Varying (LTV) system into a contractive Linear Time Invariant (LTI) system, for which quadratic Lyapunov functions can be built to characterize its contraction region. The event-triggered stability condition is defined on this form, and then is translated to the original variables. We can then extend the proposed approach to control a certain class of affine nonlinear systems, through a local in time linearization. The main novelties of the proposed approach can be summed up in the following points:

- We design a novel method based on contraction analysis to derive event-triggering conditions for LTV systems, and which can be extended to a certain class of nonlinear systems. Contraction analysis offers a way to transform an LTV system into an LTI system. Therefore, we provide a detailed approach to design the event-triggered control with optimal parameters.

- We demonstrate the design of two types of event-triggering strategies that can be adapted to several types of systems, or to several stages of the control process to achieve better performance.

- We provide a detailed proof of the stability of both strategies. We also build on previous work to demonstrate the existence of a minimum inter-event time, avoiding thus the Zeno phenomenon.

It is worth pointing out that, in a certain way, the proposed method can be considered as an extension of the method introduced in [25] to the class of LTV systems, where the event-triggering conditions are activated to preserve the contraction property of the system in the new state coordinate and so to preserve its stability. On the other hand, the methods in [4, 15, 17, 19] deal with a larger class of dynamic systems but they assume the existence of Lyapunov functions for the systems. This assumption is not always easy to satisfy even for LTV systems. Thus, the advantage of the proposed method consists in providing a systematic approach to co-design the control law and its event-triggering condition that ensure the stability of the considered class of systems.

In Section 2 we recall how to design state feedback gains that ensure the contraction property in the context of linear time varying systems. Section 3 is devoted to the definition of the proposed event-triggering condition. We show that it leads to a stable and Zeno phenomenon free event-controlled system. The specific case of two-dimensional systems is addressed in Section 4, where we give the explicit form of all the matrices involved in the control design step and two case studies are considered. In each case the continuous and event-triggered situations are compared. We extend the proposed approach to nonlinear systems and illustrate with a two-dimensional example.

2 Controlling Linear Time Varying systems via Contraction Analysis

We consider the stabilization of a LTV system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + B(t)u(t), \qquad t > 0 \\ \mathbf{x}(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{1}$$

with a state feedback control law defined by

$$u(t) = K(t)\mathbf{x}(t). \tag{2}$$

We suppose that $\mathbf{x}(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$ for all $t > 0$. Hence $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times 1}$, and $K(t) \in \mathbb{R}^{1\times n}$.

Notation. We denote by λ_M and Λ_M the smallest and greatest (real) eigenvalues of a symmetric definite positive matrix M.

2.1 Towards a Linear Time Invariant system

If we apply to system (1)-(2) a time-dependent change of coordinate $\mathbf{z}(t) = \Theta(t)\mathbf{x}(t)$, we can compute the time evolution of $\mathbf{z}(t)$:

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \dot{\Theta}(t)\mathbf{x}(t) + \Theta(t)\dot{\mathbf{x}}(t) \\ &= \dot{\Theta}(t)\mathbf{x}(t) + \Theta(t)\big(A(t)\mathbf{x}(t) + B(t)K(t)\mathbf{x}(t)\big) \\ &= \Big(\dot{\Theta}(t) + \Theta(t)(A(t) + B(t)K(t))\Big)\,\Theta^{-1}(t)\mathbf{z}(t). \end{aligned}$$

We want to choose $\Theta(t)$ and $K(t)$ such that this system governing the time evolution of $\mathbf{z}(t)$ is time-invariant and stable. More precisely, we would like to simply have

$$\dot{\mathbf{z}}(t) = F\mathbf{z}(t), \tag{3}$$

where F is a constant Frobenius companion matrix:

$$F = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_1 & -f_2 & -f_3 & \dots & -f_n \end{pmatrix}.$$

This means that $\Theta(t)$ should be solution to

$$F\Theta(t) = \dot{\Theta}(t) + \Theta(t)\Big(A(t) + B(t)K(t)\Big). \tag{4}$$

There are potentially many solutions to Equation (4) and we make below specific choices.

2.2 Constructing the change of variable

The choice of a Frobenius matrix for F implies that, denoting θ_j the jth row in Θ and $f = (f_1, \ldots, f_n)$, we can simply write (4) as

$$\begin{aligned} \dot{\theta}_j(t) + \theta_j(t)(A(t) + B(t)K(t)) &= \theta_{j+1}(t), \qquad j = 1, \ldots, n-1, \\ \dot{\theta}_n(t) + \theta_n(t)(A(t) + B(t)K(t)) &= -f\Theta(t). \end{aligned}$$

We moreover want the change of variable $\Theta(t)$ not to depend on the control. To this aim we prescribe

$$\Theta(t)B(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d(t) \end{pmatrix} \equiv D(t)$$

and we therefore have to solve

$$\dot{\theta}_j(t) + \theta_j(t)A(t) = \theta_{j+1}(t), \qquad j = 1, \ldots, n-1, \tag{5}$$

$$\dot{\theta}_n(t) + \theta_n(t)A(t) + d(t)K(t) = -f\Theta(t). \tag{6}$$

Then (5) allows to construct by induction all the rows given θ_1, and (6) yields the feedback gain matrix $K(t)$:

$$K(t) = -\frac{f\Theta(t) + \dot{\theta}_n(t) + \theta_n(t)A(t)}{d(t)}. \tag{7}$$

Now we want to find convenient $\theta_1(t)$ and $d(t)$, and find a condition under which we can ensure that $d(t) \neq 0$. To accomplish this, we multiply equations (5) and (6) by $B(t)$ and define the Lie derivatives

$$L^j B(t) = A(t)L^{j-1}B(t) - \frac{d}{dt}\left(L^{j-1}B(t)\right).$$

This yields respectively

$$\begin{aligned} \theta_1(t)L^j B(t) &= 0, \qquad j = 0, \ldots, n-2, \\ \theta_1(t)L^{n-1}B(t) &= d(t). \end{aligned}$$

The details can be found in Appendix A. The $L^j B(t)$ are column vectors, and we gather then in a matrix $\mathcal{B}(t) \in \mathbb{R}^{n\times n}$ and the previous conditions are cast as

$$\mathcal{B}(t)^T \theta_1(t)^T = D(t). \tag{8}$$

If $\det \mathcal{B}(t) \neq 0$, this system admits a unique solution $\theta_1(t)$. Else we should choose $\theta_1^T(t) \in \mathrm{Ker}(\mathcal{B}(t)^T)$. It is worth pointing out that, in any case, $D(t)$ and $\det \mathcal{B}(t)$ vanish simultaneously. A way to ensure this is to choose $d(t) = \det \mathcal{B}(t)$. The condition to be able to define the feedback gain matrix $K(t)$ with this procedure is that $\det \mathcal{B}(t) \neq 0$. This is exactly the classical condition for the controllability of a LTV system [18].

Besides the computation of $L^{n-1}B(t)$, and therefore $\theta_1(t)$, involves derivatives of B up to order $n-1$ and A up to order $n-2$. Then we reconstruct the remaining lines in $\Theta(t)$ using equation (5) $n-1$ times. Hence, in the general case, $\Theta(t)$ can involve derivatives of B up to order $2n-2$ and A up to order $2n-3$.

Condition 1.

(a) To be able to define the state feedback gain by the analysis above, we need that $\det \mathcal{B}(t) \neq 0$ for all times t.

(b) For Θ to be continuous with respect to time, we need B to be $\mathcal{C}^{2n-2}(\mathbb{R}^+)$ and A to be $\mathcal{C}^{2n-3}(\mathbb{R}^+)$.

2.3 Continuous stability issues

Lemma 1. If F is Hurwitz, system (3) is asymptotically stable.

This result is classical in the case of linear systems. We can be more precise, namely for any symmetric positive definite matrix Q, there exists a unique symmetric positive definite matrix P, such that $F^T P + PF = -Q$ and a Lyapunov function $V(\mathbf{z}) = \mathbf{z}^T P \mathbf{z}$ such that

$$\frac{d}{dt}(\mathbf{z}^T(t) P \mathbf{z}(t)) = -\mathbf{z}^T(t) Q \mathbf{z}(t).$$

From the eigenvalues of P and Q, we can deduce that

$$\frac{d}{dt}(\mathbf{z}^T(t) P \mathbf{z}(t)) \leq -\frac{\lambda_Q}{\Lambda_P} \mathbf{z}^T(t) P \mathbf{z}(t). \tag{9}$$

We will denote $\zeta = \lambda_Q / \Lambda_P$ in the sequel. We can transpose (9) into the original domain defining $N(t) = \Theta(t)^T P \Theta(t)$, and the $N(t)$-norm defined by $\|\mathbf{x}\|^2_{N(t)} = \mathbf{x}^T N(t) \mathbf{x}$:

$$\frac{d}{dt}\|\mathbf{x}(t)\|^2_{N(t)} \leq -\zeta \|\mathbf{x}(t)\|^2_{N(t)}. \tag{10}$$

Remark 1. Although ζ is a constant coefficient, we cannot immediately deduce global asymptotic stability from Equation (10) because the norm is changing over time.

Theorem 1. Under Conditions 1 (a) and (b) and if $e^{-\zeta t}/\lambda_{N(t)} \to 0$ as $t \to +\infty$, the solution to (1) is asymptotically stable.

Proof. If $\det \Theta(t) \neq 0$, then $N(t)$ is a symmetric definite positive matrix. We can estimate the $N(t)$-norm from below by

$$\|\mathbf{x}\|_{N(t)}^2 \geq \lambda_{N(t)} \mathbf{x}^T \mathbf{x}.$$

Under Condition 1 (a) and (b) on a time interval $[0, T]$ on which $\det \Theta(t) \neq 0$, $\lambda_{N(t)}$ is a continuous function and its minimum over the interval is $\nu_T > 0$. This together with the inequality (10) leads to the estimate

$$\nu_T \mathbf{x}^T \mathbf{x} \leq \lambda_{N(t)} \mathbf{x}^T \mathbf{x} \leq \|\mathbf{x}(t)\|_{N(t)}^2 \leq \|\mathbf{x}(0)\|_{N(0)}^2 e^{-\zeta t}. \tag{11}$$

If $e^{-\zeta t}/\lambda_{N(t)} \to 0$ as $t \to +\infty$, we can conclude that the system is asymptotically stable. This is in particular true, if we can bound $\lambda_{N(t)}$ from below on $[0, +\infty)$: $\lambda_{N(t)} \geq \nu_{\min} > 0$. □

3 Event-triggered Control

3.1 Definition of triggering conditions

In event-triggered control, the control is updated only when an event occurs. Between two events the control is kept constant. Let τ_k, $k \in \mathbb{N}$, be the sequence of successive event times. Equation (1) is replaced by

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)u_k, \; \tau_k \leq t < \tau_{k+1} \tag{12}$$

with a piecewise constant state feedback control law $u_k = K(\tau_k)\mathbf{x}(\tau_k)$.

We consider two event-triggering strategies for which the triggering times are defined by induction.

3.1.1 Triggering condition based on (10)

Although estimate (10) is not a sharp one, since the control is an approximate control, we cannot expect to have $\frac{d}{dt}\|\mathbf{x}(t)\|_{N(t)}^2 \leq -\zeta \|\mathbf{x}(t)\|_{N(t)}^2$. We therefore choose a parameter $0 \leq \alpha < \zeta$ and the condition

$$\frac{d}{dt}\|\mathbf{x}(t)\|_{N(t)}^2 \leq -\alpha \|\mathbf{x}(t)\|_{N(t)}^2 \tag{13}$$

is less restricting. Capturing times when the condition (13) is violated yield the triggering times.

Strategy 1 (Decreasing norm)**.** The control law is updated at times τ_k such that

$$\tau_{k+1} = \inf \left\{ t > \tau_k \text{ such that } \frac{d}{dt}\|\mathbf{x}(t)\|^2_{N(t)} \geq -\alpha \|\mathbf{x}(t)\|^2_{N(t)} \right\}. \tag{14}$$

We notice that the choice of $\alpha = 0$ in Strategy 1 only amounts at imposing that $\|\mathbf{x}(t)\|_{N(t)}$ is decreasing.

This triggering condition involves a time derivative which is not available in practical applications. In subsection 3.3, we propose an equivalent expression to evaluate this event-triggered condition with no need to compute the time derivative of the norm of the state vector.

3.1.2 Triggering condition based on (11)

Another strategy is the one used in [25]. It is based on estimate (11) and we choose to update the control when this estimate is violated.

Strategy 2 (Exponential decay)**.** The control law is updated at times τ_k such that

$$\tau_{k+1} = \inf \left\{ t > \tau_k \text{ such that } \|\mathbf{x}(t)\|^2_{N(t)} \geq \|\mathbf{x}(0)\|^2_{N(0)} e^{-\zeta t} \right\}. \tag{15}$$

Since they only differ in the triggering condition, both strategies share the same block diagram representation given in Figure 1.

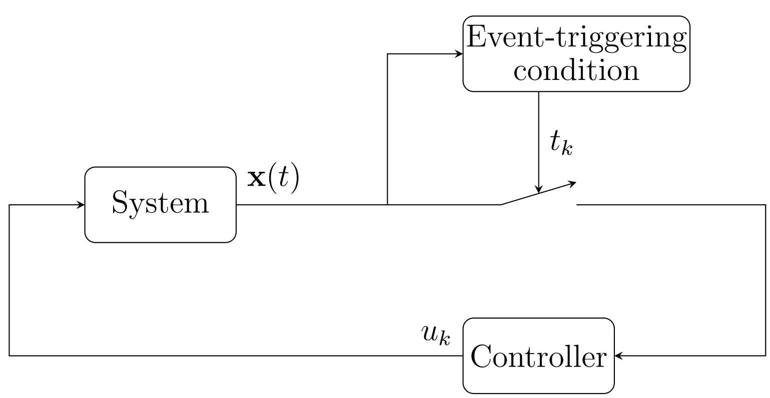


Figure 1: Block diagram of the proposed event-based control.

The structure of the underlying algorithm is therefore also the same. We provide it here for Strategy 2. Strategy 1 would only consist in replacing the if condition by $\frac{d}{dt}\|\mathbf{x}(t)\|^2_{N(t)} \geq -\alpha\|\mathbf{x}(t)\|^2_{N(t)}$.

These event based strategies have been studied in [17] and [25]. Moreover they are not in competition and can be combined. Indeed in [25] it is shown that it is better to start with Strategy 2 and then adopt Strategy 1 with $\alpha = 0$ once $\|\mathbf{x}\|_{N(t)}$ is below some threshold.

Algorithm 1 Event-based sampling (Strategy 2)

Require: A, B, $\mathbf{x}_0$, $K(t)$, $N(t)$, ζ, T

$u_k = K(t_0)\mathbf{x}_0$
while $t < T$ **do**
  $\mathbf{x}(t) = \textbf{Sensor}(\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu_k)$
  if $\|\mathbf{x}(t)\|^2_{N(t)} \geq \|\mathbf{x}(0)\|^2_{N(0)} e^{-\zeta t}$ **then**
    $u_k = K(t)\mathbf{x}(t)$
  end if
end while

Remark 2. Notice that Strategy 2 is a dynamic event-triggered mechanism that can be implemented in the following way:

$$\dot{\nu}(t) = -\zeta \nu(t) \text{ with } \nu(0) = \|\mathbf{x}(0)\|_{N(0)}$$

where $\nu(t) = \|\mathbf{x}(0)\|_{N(0)} e^{-\zeta t}$.

An event-triggering strategy is relevant if it leads to a stable controlled system, and if a minimum inter-event time is guaranteed, ensuring that there is no Zeno phenomenon. We discuss these two issues in the following paragraphs.

3.2 Stability

Theorem 2. Under Strategies 1 or 2 and for the state feedback gain defined by (7), system (12) is asymptotically stable.

Proof. The proof needs no specific knowledge on the system itself apart from the fact that it is possible to construct a feedback matrix $K(t)$ and definite positive matrices $N(t)$ for all times.

- $t = \tau_k$
  When the control is updated, the time derivative of the state vector is exactly
  $$\dot{\mathbf{x}}(t) = \big(A(t) + B(t)K(t)\big)\mathbf{x}(t)$$
  and the computations of Section 2 allow to state that
  $$\frac{d}{dt}\|\mathbf{x}(t)\|^2_{N(t)} \leq -\zeta \|\mathbf{x}(t)\|^2_{N(t)} < -\alpha \|\mathbf{x}(t)\|^2_{N(t)}.$$

- $t \in (\tau_k, \tau_{k+1})$ for Strategy 1
  Since no event is triggered between times τ_k and τ_{k+1}, this means that the condition for event-triggering is never fulfilled and
  $$\frac{d}{dt}\|\mathbf{x}(t)\|^2_{N(t)} < -\alpha \|\mathbf{x}(t)\|^2_{N(t)}, \ \forall t \in (\tau_k, \tau_{k+1}).$$

For $\alpha \geq 0$ this means that the quantity $\|\mathbf{x}(t)\|_{N(t)}$ is decreasing. More precisely for all $t \in (\tau_k, \tau_{k+1})$

$$\|\mathbf{x}(t)\|^2_{N(t)} \leq \|\mathbf{x}(\tau_k)\|^2_{N(\tau_k)} e^{-\alpha(t-\tau_k)}.$$

- $t \in (\tau_k, \tau_{k+1})$ for Strategy 2
  Since no event is triggered between times τ_k and τ_{k+1}, this means that the condition for event-triggering is never fulfilled and

$$\|\mathbf{x}(t)\|^2_{N(t)} \leq \|\mathbf{x}(0)\|^2_{N(0)} e^{-\zeta t}.$$

  Under the same conditions as for Theorem 1, $\|\mathbf{x}(t)\|$ tends to zero as times goes to infinity.

□

3.3 Minimum inter-event time

To prove that there exists a minimum inter-event time, we can refer to existing literature. Indeed for both our strategies the triggering condition, and therefore the triggering times only depend on the quantity $\|\mathbf{x}(t)\|^2_{N(t)}$ which is constructed to be equal to $\mathbf{z}(t)^T P\mathbf{z}(t)$.

Therefore the triggering conditions for Strategies 1 and 2 can be expressed entirely in terms of the $\mathbf{z}(t)$ variable as

$$\tau_{k+1} = \inf \left\{ t > \tau_k \text{ such that } \frac{d}{dt}\mathbf{z}(t)^T P\mathbf{z}(t)) \geq -\alpha \mathbf{z}(t)^T P\mathbf{z}(t) \right\},$$

and

$$\tau_{k+1} = \inf \left\{ t > \tau_k \text{ such that } \mathbf{z}(t)^T P\mathbf{z}(t) \geq \mathbf{z}(0)^T P\mathbf{z}(0) e^{-\zeta t} \right\},$$

where $\dot{\mathbf{z}}(t) = F\mathbf{z}(t)$. The construction of the triggering times is therefore the same as for the very classical case of a LTI system, and has been already studied in [17] and [25] respectively.

For practical applications, it is impossible to compare with an extremely low threshold. So, similarly to [25], we fix a time T_{lim} after which the triggering condition is simply

$$\tau_{k+1} = \inf \left\{ t > \tau_k \text{ such that } \mathbf{z}(t)^T P\mathbf{z}(t) \geq \delta \right\},$$

the time T_{lim} being defined as the first time for which $\mathbf{z}(t)^T P\mathbf{z}(t) = \delta$ for a predefined δ. This is called Strategy 3 in the proof.

According to the literature on the LTI case, we can state the following theorem, for which we give a proof in Appendix C.

Theorem 3. Suppose Strategies 1 or 2 until time T_{lim} and then a constant threshold is used, then for the state feedback gain defined by (7), system (12) has no Zeno phenomenon.

Remark 3. It is worth pointing out that, in practice it is recommended to add constant positive value to the decreasing threshold of Strategy 2 to avoid possible events related to effect of the system noise and disturbance. To avoid the effect of disturbances in the numerical evaluation of the time derivative in Strategy 1, it is possible to replace it by its value, namely

$$\begin{aligned}\frac{d((\mathbf{z}(t)^T P\mathbf{z}(t))}{dt} =& -\mathbf{z}(t)^T Q\mathbf{z}(t)\\ &+ 2\mathbf{z}(t)^T PB(t)^T \left(K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(t)\Theta^{-1}(t)\mathbf{z}(t)\right),\end{aligned}$$

which only involves the state and not its derivative.

4 Examples in the two-dimensional case

4.1 Computation of the feedback matrix

The derivation of the state feedback matrix in the two-dimensional case is detailed in Appendix B.1. Under the assumption $d \neq 0$, we obtain for θ_1 a vector which does not dependent on A and is orthogonal to B, namely $\theta_1 = \begin{pmatrix}-b_2 & b_1\end{pmatrix}$. In the next example, we will always consider that the control is simply added to the second equation, which corresponds to $B(t) \equiv \begin{pmatrix}0 & 1\end{pmatrix}^T$. In this very specific case $\det \mathcal{B}(t) = -a_{12}(t)$, which yields a simple characterization of the ability to construct a change of variable for all t and a feedback matrix $K(t)$.

We will also use for F the matrix

$$F = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$$

which eigenvalues are -2 and -3. The optimum value for $\zeta = \lambda_Q/\Lambda_P$ is obtained when Q is proportional to the identity matrix I. In this case, for $Q = qI$, the solution to the Lyapunov equation $F^T P + PF = -Q$ is given by

$$P = \frac{q}{2f_1 f_2}\begin{pmatrix} f_1(f_1+1) + f_2^2 & f_2 \\ f_2 & f_1 + 1 \end{pmatrix}.$$

Here we choose $Q = 3I$ and

$$P = \frac{1}{10}\begin{pmatrix} 67 & 5 \\ 5 & 7 \end{pmatrix}.$$

Since $\det(P) = 4.44$, we will simply have $\det N(t) = 4.44 \det^2 \Theta(t)$. We also compute $\zeta \simeq 0.89$.

For each example, we can compare numerical simulations of the continuous control and event-triggered controls. For continuous control, a sufficiently fine time-step is considered to mimic continuity. Since the equations we consider are not very stiff, we use a simple explicit Euler scheme for the computations.

In event-triggered control, there are many ways to parametrize the algorithm. The choice of the time-step would be important for applications. It is the frequency at which we consider updating the control. Since we want a few updates, having a relatively large time-step is an option, but this means missing the exact event times at this scale also. Since we use an Euler scheme, the control is computed using the previous state, and the system can violate the constraints for the duration of a time-step, which can be harmful to the operation of the system, but can also be an advantage as we will see in examples. The two event-triggering strategies are compared, and in the first strategy three different values of α are used, namely $\alpha = 0$, $\alpha = \zeta/2$, and $\alpha = \zeta$.

Remark 4. The matrix F we have chosen here has the particularity to have real negative eigenvalues. So an alternate construction of the Lyapunov function could have been to set $V(\mathbf{z}) = \mathbf{z}^T C^T C \mathbf{z}$, where C is a change-of-basis matrix. This is detailed in Appendix B.2. In our example it yields better results since the decreasing rate ζ is bigger. But since it is not possible to extend this to more general matrices, we have chosen the general approach of the Lyapunov equation in this paper.

4.2 Numerical implementation

To discretize the equations we use a simple Euler scheme. More elaborate, and in particular implicit methods would be a bit tricky to extend to the event-based context, since we would have to predict future events. Besides addressing stiff problems is not the purpose of the present paper.

Given a time step δt, we compute the state and control at times $t_i = i\delta t$. In the continuous control case we compute

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \delta t(A(t_i) + B(t_i)K(t_i))\mathbf{x}_i.$$

In the event-based case, the event triggering condition is computed at time t_i, for Strategy 1

$$\frac{\|\mathbf{x}_i\|^2_{N(t_i)} - \|\mathbf{x}_{i-1}\|^2_{N(t_{i-1})}}{\delta t} \geq -\alpha \|\mathbf{x}_i\|^2_{N(t_i)},$$

and for Strategy 2

$$\|\mathbf{x}_i\|^2_{N(t_i)} \geq \|\mathbf{x}_0\|^2_{N(0)} e^{-\zeta t_i}.$$

If a new event is triggered, the control is updated as $\bar{u} = K(t_i)\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ is computed as

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \delta t \left(A(t_i)\mathbf{x}_i + B(t_i)\bar{u} \right).$$

There are two parameters that can be tuned to perform the numerical simulations: the α if Strategy 1 is chosen which is in the range $[0, \zeta]$, and the time step δt.

4.3 A first example — polynomial matrix

As a first example, we study

$$\begin{cases} \dot{x}_1(t) &= t x_1(t) + x_2(t), \\ \dot{x}_2(t) &= t^2 x_2(t) + u(t), \end{cases}$$

where the associated matrix $A(t)$ is polynomial in the time variable and for which $d(t) = -1 \neq 0$ for all time, and we can derive

$$\Theta(t) = \begin{pmatrix} -1 & 0 \\ -t & -1 \end{pmatrix}, \; K(t) = -\begin{pmatrix} 7 + 5t + t^2 & 5 + t + t^2 \end{pmatrix}.$$

The norm matrix in the original domain is

$$N(t) = \frac{1}{10} \begin{pmatrix} 67 + 10t + 7t^2 & 5 + 7t \\ 5 + 7t & 7 \end{pmatrix}.$$

On Figure 2, we compare a continuous control and event-triggered control for $\alpha = \zeta$ on this test-case for a fine time-step $\delta t = 10^{-4}$. For each type of control we plot the time evolution of x_1 and x_2 (top left) and observe that both strategies do indeed succeed in stabilizing the trajectory. We also plot the control u (top right), the usual Euclidian norm of x (bottom left) and its $N(t)$-norm (bottom right).

The goal in implementing an event-triggered control is to have less updates of the control. Is it really the case? To discuss this, we plot on Figure 3 the number of updates with respect to the time step for the two event-triggering strategies and various values of α. We observe that for Strategy 1 the number of updates stabilize to some limit value as δt becomes small, which is the theoretical number of updates needed (on the considered time interval $[0, 10]$) for the continuous equation. This means that for these values of the time-step the numerical discretization has more or less no impact on the control updates. Strategy 2 based on the exponential decay leads to much larger values of the number of updates.
For large time steps, we see that the event-triggered method yields a low number of updates which is very close for all values of α except when α is too close to ζ. This is explained by the fact that event are not captured precisely enough for large time steps.

Now we may want to know whether the updates occur regularly over time. Figure 4 gathers two representations of the control updates: the times of the updates are given as gray spikes, and the cumulative value of the number of updates is drawn in blue, which makes it easier to show the density of updates when the spike pack is dense. In this first test case the updates do occur regularly over time since there is no plateau in the cumulative values of the updates displayed on Figure 4. However, when the time-step becomes small, the number of updates is more than linear (quadratic on this example), which may alter the performances of the algorithm.

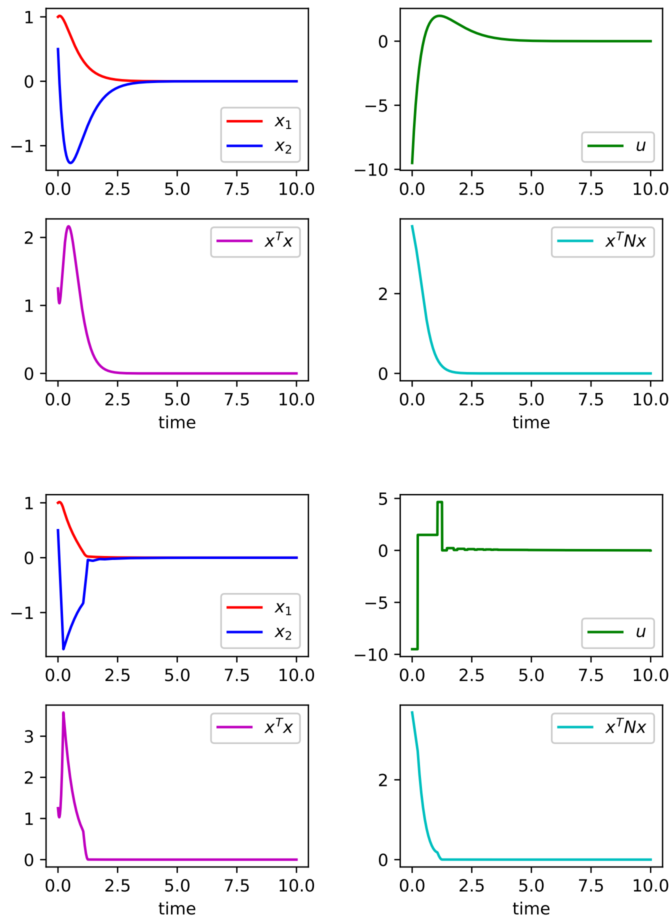


Figure 2: Polynomial matrix case: Comparison of continuous (top) and event-triggered ($\alpha = \zeta$, bottom) controls for $x_0 = (1\ .5)^T$ and for $\delta t = 10^{-4}$.

4.4 A degenerate example — rotating matrix case

Let us now consider

$$\begin{cases} \dot{x}_1(t) &= \cos t\ x_1(t) + \sin t\ x_2(t), \\ \dot{x}_2(t) &= -\sin t\ x_1(t) + \cos t\ x_2(t) + u(t), \end{cases}$$

Here $A(t)$ is a rotation matrix, which certainly will involve regular updates of the control, and the case is degenerate since $d(t) = -\sin t$ which vanishes for $t = k\pi$, $k \in \mathbb{N}$. However we can compute

$$\Theta(t) = \begin{pmatrix} -1 & 0 \\ -\cos t & -\sin t \end{pmatrix}.$$

and for $t \neq k\pi$,

$$K(t) = \begin{pmatrix} -\frac{f_1 + f_2 \cos t - \sin t + \cos^2 t - \sin^2 t}{\sin t} & -\frac{f_2 \sin t + \cos t + 2 \cos t \sin t}{\sin t} \end{pmatrix},$$

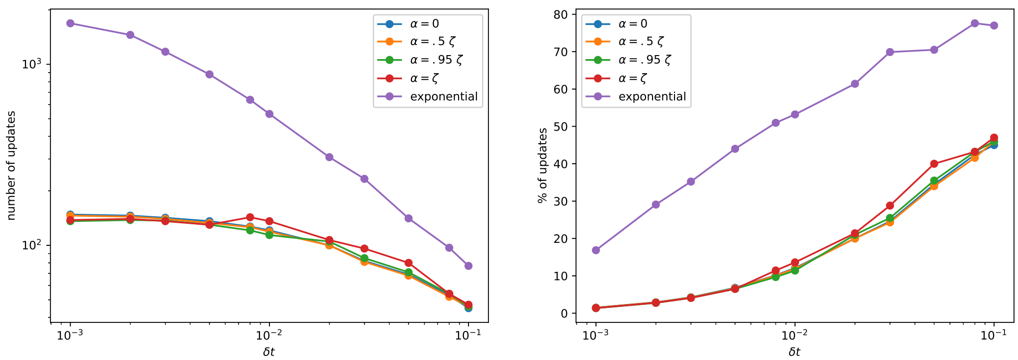
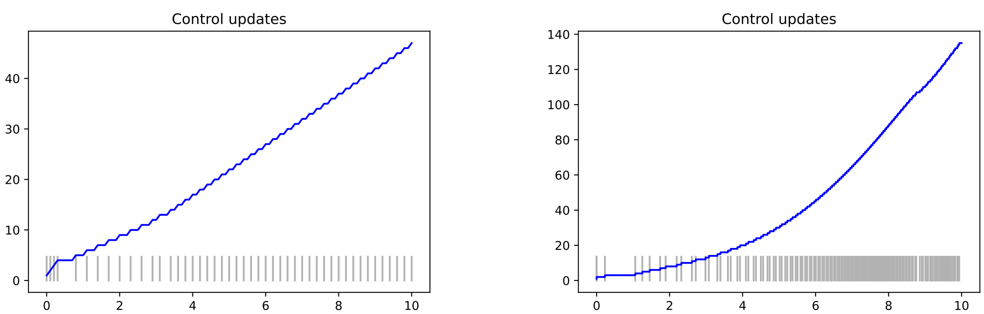


Figure 3: Polynomial matrix case: Number of updates with respect to the time-step, total value (left), percentage of the discretization times (right).



$\delta t = 10^{-1}$ $\delta t = 10^{-4}$

Figure 4: Polynomial matrix case: Times (gray) and cumulative values of the number (blue) of updates for large ($\delta t = 10^{-1}$, left) and small ($\delta t = 10^{-4}$, right) time-steps.

$$N(t) = \begin{pmatrix} 67 + 10\cos t + 7\cos^2 t & (5 + 7\cos t)\sin t \\ (5 + 7\cos t)\sin t & 7\sin^2 t \end{pmatrix}$$

and $\det N(t) = 4.44\sin^2 t$ which is not bounded from below, even on finite-length time intervals. This example therefore does not fulfill the conditions for the construction of the contraction method, since we violate Condition 1(a). We can however perform some numerical simulations, taking a time-step which avoids carefully times $t = k\pi$ (and a special treatment of $t = 0$). If the time step is relatively fine, the method leads to very large values on the controls as well as strong deviations in the trajectory, as displayed on Figure 5 for $\delta t = 10^{-4}$. Of course this behavior is not wanted, all the more as this would lead to a saturation of actuators in a practical implementation.

The goal of event-triggered control being to sample less often, what does happen if a relatively large time step is taken? This is illustrated on Figure 6 for $\delta t = 10^{-2}$. The deviation of the system is smaller ($\|x\| \simeq 60$ vs. $\simeq 1000$ for $\delta t = 10^{-4}$) and the control still takes large values near the two first

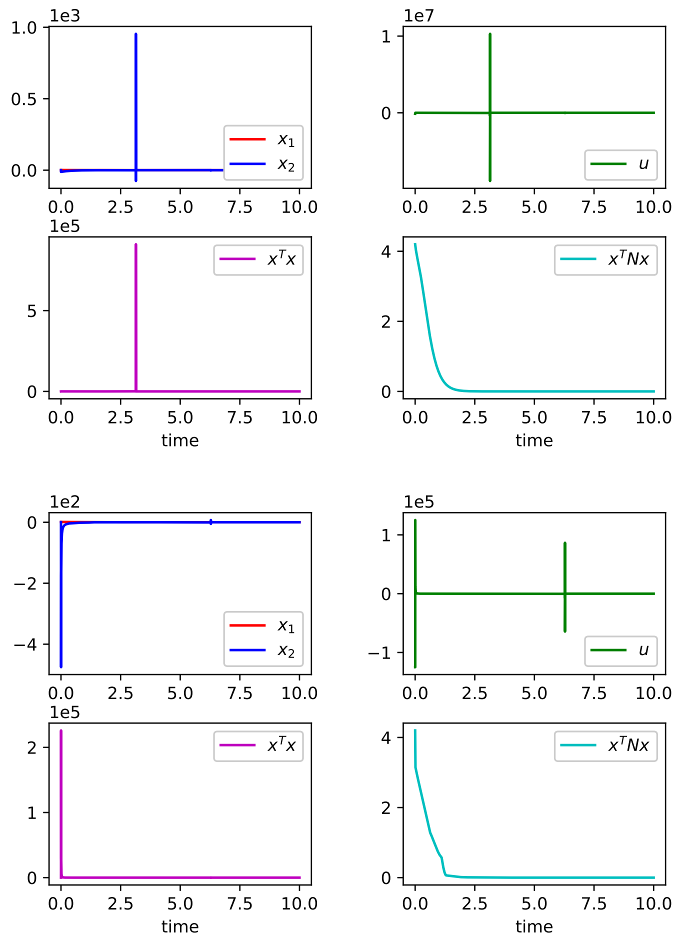


Figure 5: Rotating matrix case: Comparison of continuous (top) and event-triggered controls for $\alpha = \zeta$ (bottom) and $\delta t = 10^{-4}$.

singularities, but they are much smaller than for a smaller time step.

The event-triggering strategies can also be compared for this example and lead to a different conclusion from the previous example as can be seen on Figure 7. Indeed the number of updates does not seem to converge to a finite value as δt goes to zero. It seems on this specific example to be of order $\delta t^{-1/2}$ for all admissible values of ζ.

If we compare on Figure 8 the update times for Strategy 1 and $\alpha = \zeta$ and for small and large values of the time step, we first see that for a large time step, the updates occur relatively regularly in time. This can be explained by the fact that the matrix $A(t)$ is a rotation matrix, and the change of variable $\Theta(t)$ to a fixed basis has to be updated regularly. When the time step is smaller, besides these regular updates, there are successive updates in the vicinity of the degeneracies, which degrades the performances of the event-triggered approach in terms of reduction of the number of updates.

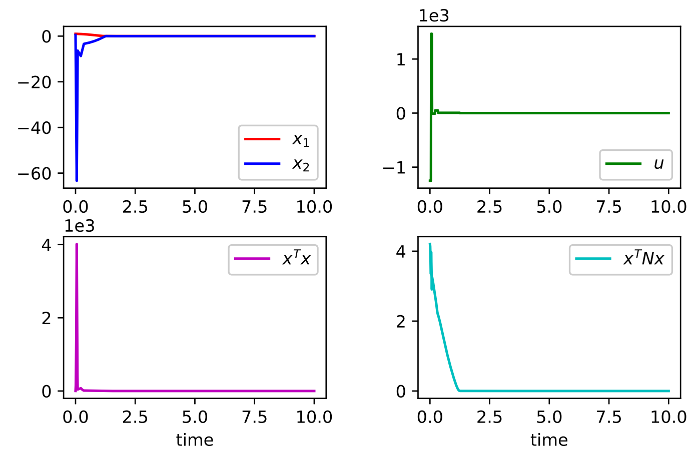


Figure 6: Rotating matrix case: Event-triggered controls for $\alpha = \zeta$ and and a large time-step $\delta t = 10^{-2}$.

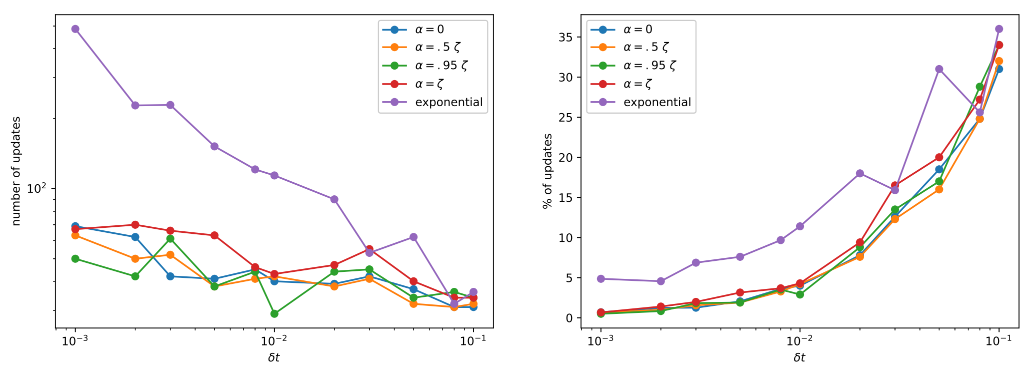


Figure 7: Rotating matrix case: Number of updates with respect to the time-step.

5 Extension to nonlinear systems

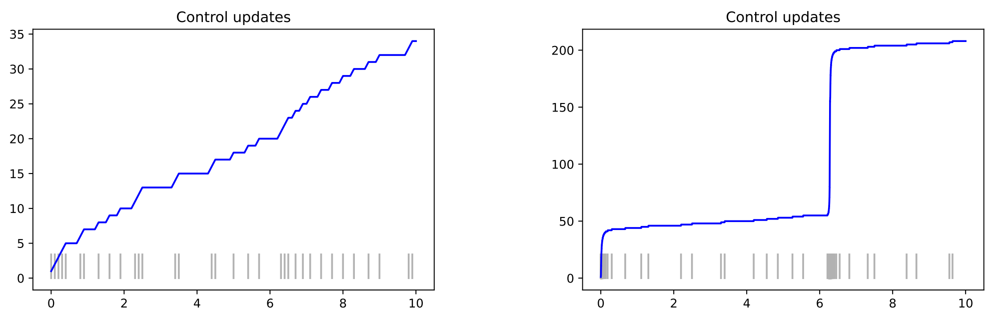
We now consider the stabilization of control affine nonlinear systems

$$\begin{aligned} \dot{\mathbf{x}}(t) &= F(t, \mathbf{x}(t)) + B(t)u(t), \qquad t > 0 \\ \mathbf{x}(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{16}$$

with a state feedback control law $u(t) = K(t, \mathbf{x}(t))\mathbf{x}(t)$, and where $F : \mathbb{R} \times \mathbb{R}^n$ is supposed to be differentiable with respect to its second variable. Locally in time, the system can be linearized as

$$\dot{\mathbf{y}}(t) = \nabla_x F(t, \mathbf{x}(t))\mathbf{y}(t) + B(t)u(t),$$

and denoting $A(t) \equiv \nabla_x F(t, \mathbf{x}(t)) \in \mathcal{M}_{n,n}(\mathbb{R})$, this casts the system at time t in the same form as (1). From matrices $A(t)$ and $B(t)$, we can derive the



$\delta t = 10^{-1}$ $\delta t = 10^{-4}$

Figure 8: Rotating matrix case: Times (gray) and cumulative values of the number (blue) of updates for large ($\delta t = 10^{-1}$, left) and small ($\delta t = 10^{-4}$, right) time-steps.

feedback matrix as in Section 2, and the construction of matrices $\Theta(t)$ and $K(t)$ follow the same scheme.

5.1 Second order scalar equations

A particular case of this is the case of second order equations of the type

$$\ddot{x}(t) = g(x(t), \dot{x}(t)) + u.$$

Writing this as a first order system for the variables $(x_1, x_2) = (x, \dot{x})$, we have

$$\begin{aligned} \dot{x}_1(t) &= x_2, \\ \dot{x}_2(t) &= g(x_1(t), x_2(t)) + u. \end{aligned}$$

This system falls in the $B \equiv \begin{pmatrix} 0 & 1 \end{pmatrix}^T$-case which is developed in Appendix B.1, and we have $a_{11} = 0$, $a_{12} = 1$, $a_{21} = \partial_{x_1} g(x_1(t), x_2(t))$, and $a_{22} = \partial_{x_2} g(x_1(t), x_2(t))$. Then we simply have $\Theta(t) = -I$ for all time, and $\det \mathcal{B} = -1$. This makes the application of the contraction method to such systems not very interesting. Note that we have

$$K = -\begin{pmatrix} f_1 + \partial_{x_1} g(x_1(t), x_2(t)) & f_2 + \partial_{x_2} g(x_1(t), x_2(t)) \end{pmatrix}.$$

For the numerical example, we use a test case from [1, 22], namely a robotic manipulator system, governed by the pendulum equation

$$J\ddot{q}(t) + B\dot{q} + MgL \sin q(t) = u(t),$$

with $MgL = 10$, $J = 1$ and $B = 2$ and initial data $q(0) = 20$, and $\dot{q}(0) = 0$. This correspond to $g(x_1, x_2) = -\frac{MgL}{J} \sin x_1 - \frac{B}{J} x_2 = -10 \sin x_1 - 2x_2$.The state feedback matrix reads

$$K(t) = -\begin{pmatrix} 6 - 10 \cos x_2(t) & 3 \end{pmatrix}.$$

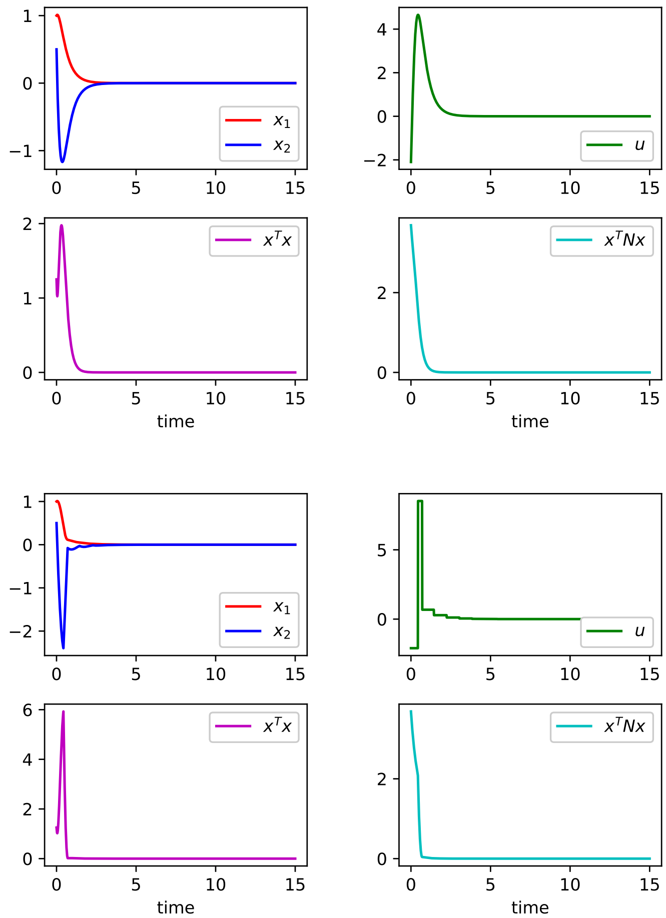


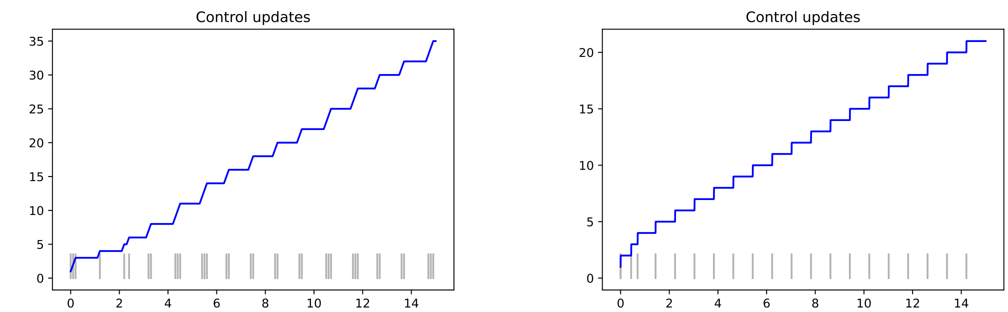
Figure 9: Pendulum case: Comparison of continuous and event-triggered controls for $\alpha = \zeta$ and $\delta t = 10^{-4}$.

The norm matrix in the original domain is $N(t) = P$.

On Figure 9, we compare as previouly a continuous control and event-triggered control for $\alpha = \zeta$ on this test-case for a fine time-step $\delta t = 10^{-4}$.

This system is not at all singular and behaves smoothly, much smoother than the two previous example in fact since the coefficients remain bounded and there is no singularity to avoid. Therefore as Figure 10 shows, there is no advantage in taking large simulation time steps, and the number of updates is smaller for a small time step than for a large one. In this context, talking large time step means determining too coarsely the event times. With a smaller time step the events are captured on time, and the most appropriated control is applied sooner.

The number of updates is quite similar to that of [1], as well as the time distribution of events. It is smaller than in [22] but the parameterization of the methods follows quite different principles and a fair comparison difficult.



$\delta t = 10^{-1}$ $\delta t = 10^{-4}$

Figure 10: Pendulum case: Times (gray) and cumulative values of the number (blue) of updates for large ($\delta t = 10^{-1}$, left) and small ($\delta t = 10^{-4}$, right) time-steps.

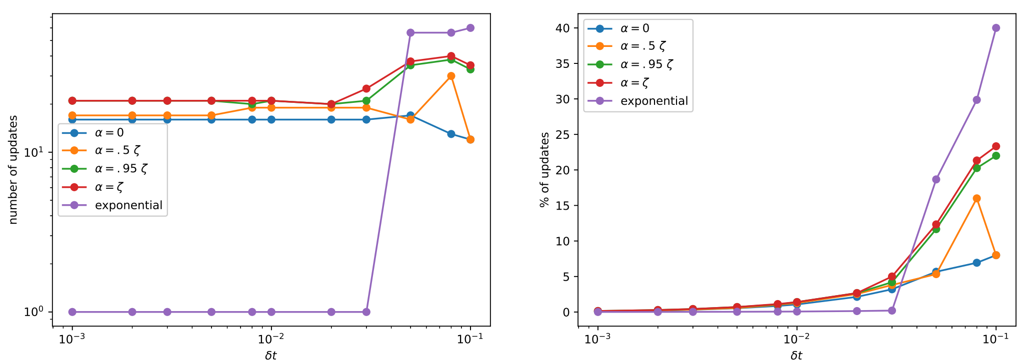


Figure 11: Pendulum case: Number of updates with respect to the time-step.

5.2 Non-trivial similarity transformation

We can also treat nonlinear systems for which the change of variable is non-trivial. Indeed we now consider the following two-state, control affine non-linear system, with one input and one output [7]

$$\begin{cases} \dot{x}_1(t) &= x_2(t) + \sin x_1(t), \\ \dot{x}_2(t) &= x_1^2(t) + u(t), \end{cases}$$

with initial data $x_0 = \begin{pmatrix} 1 & .5 \end{pmatrix}^T$. The linearization at time t of this system leads to

$$A(t) = \begin{pmatrix} \cos x_1(t) & 1 \\ 2x_1(t) & 0 \end{pmatrix}, \; B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and to the change of variable

$$\mathcal{B}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \Theta(t) = \begin{pmatrix} -1 & 0 \\ -\cos x_1(t) & -1 \end{pmatrix}$$

for which $d(t) = -1 \neq 0$ for all $t \in \mathbb{R}$. We also compute

$$K(t) = \left(-(f_1 + f_2 \cos x_1(t) - \sin x_1(t) + \cos^2 x_1(t) + 2x_1(t)) \quad -(f_2 + \cos x_1(t)) \right),$$

$$N(t) = \begin{pmatrix} 67 + 10 \cos x_1(t) + 7\cos^2 x_1(t) & 5 + 7\cos x_1(t) \\ 5 + 7\cos x_1(t) & 7 \end{pmatrix},$$

and $\det N(t) = 4.44$.

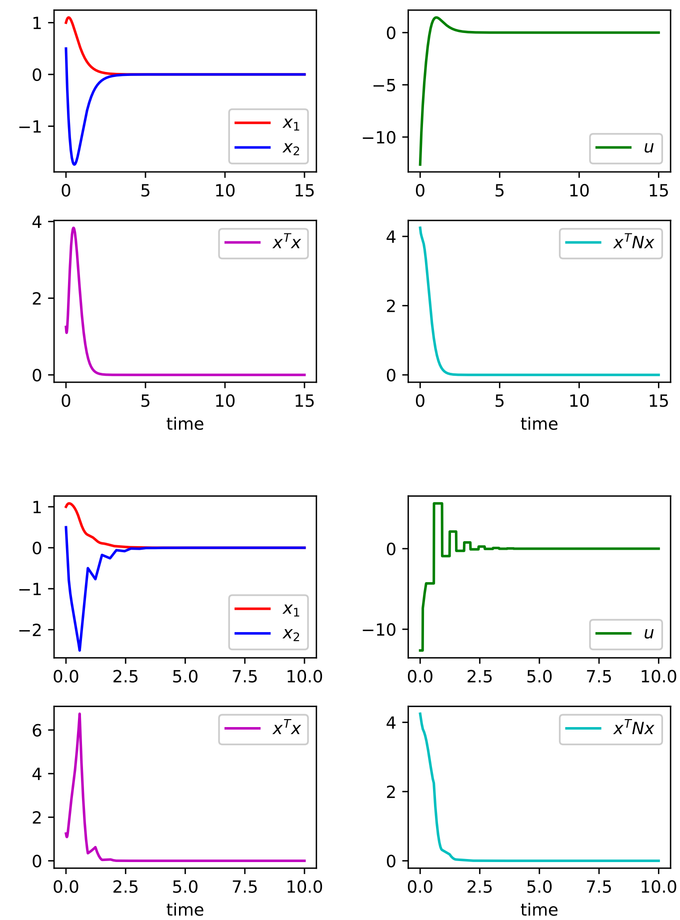


Figure 12: Control affine nonlinear system with constant B: Comparison of continuous and event-triggered controls ($\delta t = 10^{-4}$).

The first results on the comparison of the continuous and event-triggered controls are displayed on Figure 12. They yield similar results to the first example (see Figure 13) for values of α far from its maximum value: Strategy 1 still proves to be the best one, and the number of updates converges to some limit as δt goes to zero, which is the number of updates which would be needed if a continuous implementation of the event-triggered strategy were possible.

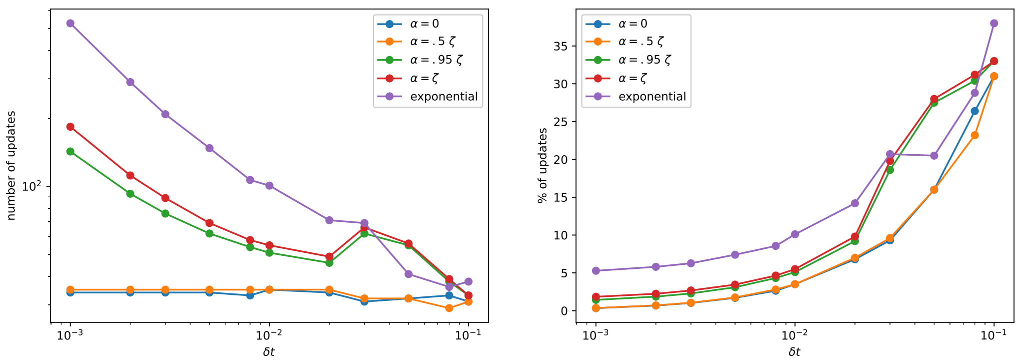
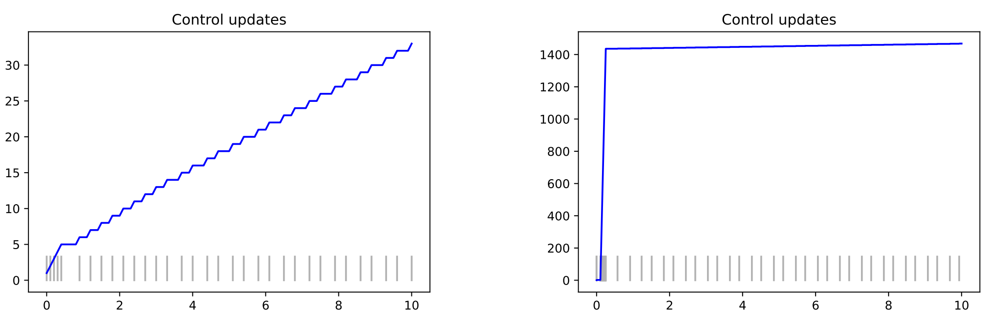


Figure 13: Control affine nonlinear system with constant B: Number of updates with respect to the time-step .

The analysis of the update time stamps (see Figure 14) shows that the control is updated regularly in time for a large time steps. For a small time step, the system undergoes a large number of control updates at the beginning of the time evolution. Then the control is satisfactory for the sequel of the time evolution (this is very different from the previous rotating case) and has not to be updated anymore.



$\delta t = 10^{-1}$ $\delta t = 10^{-4}$

Figure 14: Control affine nonlinear system with constant B: Times (gray) and cumulative values of the number (blue) of updates for large ($\delta t = 10^{-1}$, left) and small ($\delta t = 10^{-4}$, right) time-steps.

6 Conclusion

In this paper, we have introduced a step-by-step event-triggered control algorithms that can be applied to LTV systems and a class of nonlinear systems, without going through the trouble of finding a Lyapunov function. The proposed method consists in a detailed procedure that, unlike methods that rely on Lyapunov theory, contains information on how to choose the design parameters and tune them. In the meanwhile, this method leaves several degrees of freedom to the user, like the choice of the state matrix of the transformed system.

This work opens the door for many challenging research problems. For instance, it is of great interest to be able to extend this approach to deal with the case of nonlinear time-varying systems. The robustness of the proposed methods to time delays should be evaluated in future works to ensure stability of the systems even if there is a latency in the reception of the updated control value. On the other hand, it is important to extend this method to deal with the tracking problem. In fact, by considering the virtual displacement between two neighboring trajectories inside a contraction region, it is possible to design event-based set-point tracking controllers.

Acknowledgment

This work has been partially supported by the LabEx PERSYVAL-Lab (ANR-11-61 LABX-0025-01).

A Introducing Lie Derivatives

The Lie derivatives introduced in Section 2 are defined in [10] and [11]. They are used because of the following computation. Recall equations (5) multiplied by $B(t)$:

$$(\dot{\theta}_j(t) + \theta_j(t)A(t))B(t) = \theta_{j+1}(t)B(t).$$

In the $n-2$ first equations we have $\theta_{j+1}(t)B(t) = 0$, and in the last one $\theta_n(t)B(t) = d(t)$. We compute step by step

$$\begin{aligned}
0 &= \theta_1(t)B(t) = \theta_1 L^0 B(t) \text{ where } L^0 = I,\\
0 &= \theta_2(t)B(t) = (\dot{\theta}_1(t) + \theta_1(t)A(t))B(t),\\
&= (\dot{\theta}_1(t) + \theta_1(t)A(t))B(t) - \frac{d}{dt}(\theta_1(t)B(t))\\
&= \theta_1(t)A(t)B(t) - \theta_1(t)\frac{d}{dt}B(t) \text{ which allows to eliminate } \dot{\theta}_1\\
&= \theta_1(t)L^1B(t) \text{ where } L^1B(t) = AL^0B(t) - \frac{d}{dt}(L^0B(t)),\\
0 &= \theta_3(t)B(t) = (\dot{\theta}_2(t) + \theta_2(t)A(t))B(t),\\
&= (\dot{\theta}_2(t) + \theta_2(t)A(t))B(t) - \frac{d}{dt}(\theta_2(t)B(t))\\
&= \theta_2(t)A(t)B(t) - \theta_2(t)\frac{d}{dt}B(t)\\
&= \theta_2(t)L^1B(t)\\
&= (\dot{\theta}_1(t) + \theta_1(t)A(t))L^1B(t) - \frac{d}{dt}(\theta_1(t)L^1B(t))\\
&= \theta_1(t)L^2B(t) \text{ where } L^2B(t) = A(t)L^1B(t) - \frac{d}{dt}(L^1B(t)).
\end{aligned}$$

Iterating this process, and setting

$$L^jB(t) = A(t)L^{j-1}B(t) - \frac{d}{dt}(L^{j-1}B(t)),$$

we have

$$\begin{aligned}
\theta_1(t)L^jB(t) &= 0, \qquad\qquad j = 0, \dots, n-2,\\
\theta_1(t)L^{n-1}B(t) &= d(t).
\end{aligned}$$

B Explicit derivation in the 2-D case

B.1 Construction of the feedback matrix K

In the two-dimensional case, we denote (dropping the time dependence)

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Under the assumption $d = \det(\mathcal{B}) \neq 0$, we can compute

$$\begin{aligned}
\mathcal{B}^T &= \begin{pmatrix} b_1 & b_2 \\ (a_{11}b_1 + a_{12}b_2) - \dot{b}_1 & (a_{21}b_1 + a_{22}b_2) - \dot{b}_2 \end{pmatrix}, \\
\mathcal{B}(t)^{T^{-1}} &= \frac{1}{d} \begin{pmatrix} (a_{21}b_1 + a_{22}b_2) - \dot{b}_2 & -b_2 \\ -(a_{11}b_1 + a_{12}b_2) + \dot{b}_1 & b_1 \end{pmatrix}, \\
\mathcal{B}(t)^{T^{-1}} D(t)^T &= \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix}.
\end{aligned}$$

Hence we obtain for $\theta_1 = \begin{pmatrix} -b_2 & b_1 \end{pmatrix}$. Since $\theta_2 = \dot{\theta}_1 + \theta_1 A$, we deduce

$$\Theta = \begin{pmatrix} -b_2 & b_1 \\ a_{21}b_1 - a_{11}b_2 - \dot{b}_2 & a_{22}b_1 - a_{12}b_2 + \dot{b}_1 \end{pmatrix}.$$

Finally $dK = -f_1\theta_1 - f_2\theta_2 - \dot{\theta}_2 - \theta_2 A$. In this specific case we need a little less regularity than in the general case. Indeed, θ_1 involves only B and not its first derivative, and therefore Θ only involves B, its first derivative, and A.

In all our two-dimensional examples, we will always have $B \equiv \begin{pmatrix} 0 & 1 \end{pmatrix}^T$. This simplifies the above formulae, and

$$\Theta = -\begin{pmatrix} 1 & 0 \\ a_{11} & a_{12} \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} 0 & a_{12} \\ 1 & a_{22} \end{pmatrix}.$$

Thus $\det \mathcal{B} = -a_{12}$, which yields a simple characterization of the ability to construct a change of variable for all t. The feedback matrix is then computed as

$$K = -\frac{1}{a_{12}} \begin{pmatrix} f_1 + a_{11}f_2 + \dot{a}_{11} + a_{11}^2 + a_{12}a_{21} & a_{12}f_2 + \dot{a}_{12} + a_{11}a_{12} + a_{12}a_{22} \end{pmatrix}.$$

B.2 Change-of-basis approach

Let C, the matrix which columns are the eigenvectors of F associated to $\lambda_i < 0$, $i = 1, \ldots, n$, then $C^{-1}FC = \operatorname{diag} \lambda_i$. Setting $\mathbf{y} = C^{-1}\mathbf{z}$, we have

$$\dot{\mathbf{y}} = C^{-1}FC\mathbf{y}$$

and

$$\frac{d}{dt}(\mathbf{y}(t)^T\mathbf{y}(t)) = \mathbf{y}(t)^T(2 \operatorname{diag} \lambda_i)\mathbf{y}(t) \leq -2\lambda_{-F}\mathbf{y}(t)^T\mathbf{y}(t).$$

Let us set

$$M(t) = \Theta(t)^T(C^{-1})^T C^{-1}\Theta(t),$$

then we have

$$\mathbf{y}(t)^T\mathbf{y}(t) = \mathbf{x}(t)^T M(t)\mathbf{x}(t),$$

and defining the norm $\|x\|_{M(t)} = \sqrt{\mathbf{x}^T M(t)\mathbf{x}}$,

$$\frac{d}{dt}\|\mathbf{x}(t)\|_{M(t)} \leq -\lambda_{-F}\|\mathbf{x}(t)\|_{M(t)}.$$

In the two-dimensional case, we construct the transfer matrix

$$C = \begin{pmatrix} 1 & 1 \\ \lambda_- & \lambda_+ \end{pmatrix}$$

where $\lambda_\pm = -\frac{1}{2}f_2 \pm \frac{1}{2}\sqrt{f_2^2 - 4f_1}$, and

$$C^{-1} = \frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} \lambda_+ & -1 \\ -\lambda_- & 1 \end{pmatrix}$$

and

$$(C^{-1})^T C^{-1} = \frac{1}{(\lambda_+ - \lambda_-)^2}\begin{pmatrix} \lambda_-^2 + \lambda_+^2 & -(\lambda_+ + \lambda_-) \\ -(\lambda_+ + \lambda_-) & 2 \end{pmatrix}.$$

For the specific choice of matrix F

$$F = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix},$$

we have $\lambda_+ = -2$ and $\lambda_- = -3$, and

$$(C^{-1})^T C^{-1} = \begin{pmatrix} 13 & 5 \\ 5 & 2 \end{pmatrix},$$

We have seen that the value of ζ associated to the construction of the Lyapunov matrix is $\zeta \simeq 0.89$. Here the decreasing rate is $\lambda_{-F} = -\lambda_+ = 2$ and is therefore much bigger.

C Proof of the existence of a minimal inter-event time

Minimum value for $\|\mathbf{z}(t)\|$ for $t \leq T_{\text{lim}}$

For all time, and specifically at time t_k, $\lambda_P\|\mathbf{z}(t_k)\|^2 \leq \mathbf{z}(t_k)^T P\mathbf{z}(t_k) \leq \Lambda_P\|\mathbf{z}(t_k)\|^2$, and by definition of δ, $\delta \leq \mathbf{z}(t_k)^T P\mathbf{z}(t_k)$, and hence

$$\|\mathbf{z}(t_k)\|^2 \geq \delta/\Lambda_P.$$

At time t,

$$\dot{\mathbf{z}}(t) = F\mathbf{z}(t) - B(t)\left(K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(t)\Theta^{-1}(t)\mathbf{z}(t)\right),$$

In absence of degeneracy the product $K(t)\Theta^{-1}(t)$ and $\mathbf{z}(t)$ are both Lipschitz for $t \in [0, T_{\text{lim}})$ so that $\|K(t_k)\Theta^{-1}(t_k) - K(t)\Theta^{-1}(t)\| \leq L_K(t - t_k)$ and

$\|\mathbf{z}(t_k) - \mathbf{z}(t)\| \leq L_{\mathbf{z}}(t - t_k)$. Besides the integral form of the differential equation on $\mathbf{z}$ reads

$$\mathbf{z}(t) = e^{Ft}\mathbf{z}(t_k) - \int_{t_k}^{t} e^{F(t-s)}B(s)\left(K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(s)\Theta^{-1}(s)\mathbf{z}(s)\right) ds,$$

Since F is Hurwitz, we can bound $\|\mathbf{z}(t)\|$ for $t \in [t_k, t_k + \tau_0)$ (τ_0 to be determined) from below by

$$\|\mathbf{z}(t)\| \geq \|\mathbf{z}(t_k)\| - \frac{(t-t_k)^2}{2}\left(L_K \sup_{[t_k,t_k+\tau_0)} \|\mathbf{z}(s)\| + L_{\mathbf{z}} \max_{[t_k,t_k+\tau_0)} \|\!|K(s)\Theta^{-1}(s)\|\!|\right),$$

$$\sup_{[t_k,t_k+\tau_0)} \|\mathbf{z}(s)\| \geq \sqrt{\frac{\delta}{\Lambda_P}} - \frac{(t-t_k)^2}{2}\left(L_K \sup_{[t_k,t_k+\tau_0)} \|\mathbf{z}(s)\| + L_{\mathbf{z}} \max_{[0,T_{\lim})} \|\!|K(s)\Theta^{-1}(s)\|\!|\right),$$

where $\|\!|\cdot\|\!|$ is the operator norm associated to $\|\cdot\|$. For $t - t_k < \tau_0$ sufficiently small (not depending on k) , we can have $(t-t_k)^2 L_K/2 < 1$ and then we first have

$$\sup_{[t_k,t_k+\tau_0)} \|\mathbf{z}(s)\| \geq \frac{1}{2}\sqrt{\frac{\delta}{\Lambda_P}} - \frac{(t-t_k)^2}{4} L_{\mathbf{z}} \max_{[0,T_{\lim})} \|\!|K(s)\Theta^{-1}(s)\|\!|.$$

Up to a possible further reduction of τ_0,

$$\sup_{[t_k,t_k+\tau_0)} \|\mathbf{z}(s)\| \geq \frac{1}{4}\sqrt{\frac{\delta}{\Lambda_P}} \equiv \varepsilon.$$

We also have a maximum bound for $\|z(t)\|^2$ that stems from $\mathbf{z}(t)^T P\mathbf{z}(t) \leq \mathbf{z}(0)^T P\mathbf{z}(0)$, namely

$$\|z(t)\|^2 \leq \mathbf{z}(0)^T P\mathbf{z}(0)/\lambda_P \equiv Z^2.$$

Notice that this result is independent of the choice of the triggering condition. The sequel depends on the strategy.

Minimum inter-event time for Strategy 1

We have

$$\frac{d((\mathbf{z}(t)^T P\mathbf{z}(t))}{dt} = -\mathbf{z}(t)^T Q\mathbf{z}(t) + \left(K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(s)\Theta^{-1}(s)\mathbf{z}(s)\right)^T B^T P\mathbf{z}(t) + \mathbf{z}(t)^T PB(t)^T\left(K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(t)\Theta^{-1}(t)\mathbf{z}(t)\right).$$

We therefore have

$$\frac{d((\mathbf{z}(t)^T P\mathbf{z}(t))}{dt} \leq -\lambda_Q\|\mathbf{z}(t)\|^2 + 2\left\|\mathbf{z}(t)^T PB(t)^T\left(K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(t)\Theta^{-1}(t)\mathbf{z}(t)\right)\right\|.$$

To show that for a certain time $\dfrac{d((\mathbf{z}(t)^T P\mathbf{z}(t))}{dt}$ remains lower than $-\alpha\mathbf{z}^T(t)P\mathbf{z}(t)$ which is larger or equal to $-\alpha\Lambda_P\|\mathbf{z}(t)\|^2 \geq -\alpha\Lambda_P Z^2$, it is sufficient to show that

$$2\left\|\mathbf{z}(t)^T PB(t)^T\left(K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(t)\Theta^{-1}(t)\mathbf{z}(t)\right)\right\| \leq \delta(\zeta-\alpha).$$

This is valid if

$$2\left\|K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(t)\Theta^{-1}(t)\mathbf{z}(t)\right\| \leq \frac{\delta(\zeta-\alpha)}{\max\limits_{t\in[0,T_{\lim})}\|PB(t)^T\|\ Z}.$$

Now

$$\left\|K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(t)\Theta^{-1}(t)\mathbf{z}(t)\right\| \leq \sup_{[0,T_{\lim})}\|K(s)\Theta^{-1}(s)\|L_{\mathbf{z}}(t-t_k)+L_K(t-t_k)\|\mathbf{z}(t)\|.$$

Taking a sufficiently small $t-t_k < \tau_1 < \tau_0$ (independent on k) allows to have both

$$\sup_{[0,T_{\lim})}\|K(s)\Theta^{-1}(s)\|L_{\mathbf{z}}(t-t_k) \leq \frac{1}{4}\frac{\delta(\zeta-\alpha)}{\max\limits_{t\in[0,T_{\lim})}\|PB(t)^T\|\ Z},$$

$$L_K(t-t_k)\|\mathbf{z}(t)\| \leq L_K(t-t_k)Z \leq \frac{1}{4}\frac{\delta(\zeta-\alpha)}{\max\limits_{t\in[0,T_{\lim})}\|PB(t)^T\|\ Z}.$$

This value of τ_1 bounds the inter-event time by below for Strategy 1.

Minimum inter-event time for Strategy 2

We reproduce here the proof of [25] in our context. Between times t_k and t_{k+1} we know that $d(\mathbf{z}^T(t)P\mathbf{z}(t))/dt$ necessarily vanishes, therefore this quantity remains negative for a certain time.
For $t\in[t_k, t_k+\tau_0)$, $\|\mathbf{z}(t)\| \geq \varepsilon$, and therefore $-\mathbf{z}(t)^T Q\mathbf{z}(t) \leq -\lambda_Q\varepsilon^2 \equiv -\beta$.
Similarly to the proof for Strategy 1, we now want to prove that for $t-t_k$ sufficiently small

$$2\left\|\mathbf{z}(t)^T PB(t)^T\left(K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(t)\Theta^{-1}(t)\mathbf{z}(t)\right)\right\| \leq \frac{\beta}{2}.$$

This once more stems from the Lipschitz property, since for $t-t_k < \tau_2 < \tau_0$ sufficiently small, we can both have

$$\sup_{[0,T_{\lim})}\|K(s)\Theta^{-1}(s)\|L_{\mathbf{z}}(t-t_k) \leq \frac{\beta}{8\max\limits_{t\in[0,T_{\lim})}\|PB(t)^T\|\ Z},$$

$$L_K(t-t_k)\|\mathbf{z}(t)\| \leq L_K(t-t_k)Z \leq \frac{\beta}{8\max\limits_{t\in[0,T_{\lim})}\|PB(t)^T\|\ Z}.$$

This value of τ_2 bounds the inter-event time by below for Strategy 2.

Minimum inter-event time for Strategy 3

For Strategy 3, at times t_k we exactly have $\delta = \mathbf{z}(t_k)^T P\mathbf{z}(t_k)$ so the result for the minimum value of $\|\mathbf{z}(t)\|$ based on the fact that $\delta \leq \mathbf{z}(t_k)^T P\mathbf{z}(t_k)$ is still valid. As for Strategy 2, since we also have $\delta = \mathbf{z}(t_{k+1})^T P\mathbf{z}(t_{k+1})$, there is a time for which $\frac{d((\mathbf{z}(t)^T P\mathbf{z}(t))}{dt} = 0$, so we will also seek a minimum time on which this time derivative remains negative.

We once more have $-\mathbf{z}(t)^T Q\mathbf{z}(t) \leq -\lambda_Q \varepsilon^2 \equiv -\beta$, but now $\mathbf{z}(t)^T P\mathbf{z}(t) \leq \delta$, so that $\|\mathbf{z}(t)\|^2 \leq \delta/\lambda_P$. For $t - t_k < \tau_3 < \tau_0$ sufficiently small, and using a Lipschitz constant L'_K for $t \mapsto K(t)\Theta^{-1}(t)$ for $t \geq T_{\text{lim}}$, we can both have

$$\sup_{[T_{\text{lim}},\infty)} \|K(s)\Theta^{-1}(s)\| L_{\mathbf{z}}(t - t_k) \leq \frac{\beta}{8 \max\limits_{t\in[0,T_{\text{lim}})} \|PB(t)^T\| \sqrt{\delta/\lambda_P}},$$

$$L'_K(t - t_k)\|\mathbf{z}(t)\| \leq L'_K(t - t_k)\sqrt{\delta/\lambda_P} \leq \frac{\beta}{8 \max\limits_{t\in[0,T_{\text{lim}})} \|PB(t)^T\| \sqrt{\delta/\lambda_P}}.$$

which allows to bound

$$2\left\|\mathbf{z}(t)^T PB(t)^T \left(K(t_k)\Theta^{-1}(t_k)\mathbf{z}(t_k) - K(t)\Theta^{-1}(t)\mathbf{z}(t)\right)\right\| \leq \frac{\beta}{2}.$$

This value of τ_3 bounds the inter-event time by below for Strategy 3.

References

[1] Sami Al Issa, Arghya Chakravarty, and Indrani Kar. Improved event-triggered adaptive control of non-linear uncertain networked systems. *IET Control Theory and Applications*, 13(13):2146–2152, September 2019.

[2] Bruno Boisseau, John Jairo Martinez, Thibaut Raharijaona, Sylvain Durand, and Nicolas Marchand. Event-switched control design with guaranteed performances. *International Journal of Robust and Nonlinear Control*, 27(15):2492–2509, 2017.

[3] Florian D. Brunner, W.P.M.H. (Maurice) Heemels, and Franck Allgöwer. Event-triggered and self-triggered control for linear systems based on reachable sets. *Automatica*, 101:15—26, 2019.

[4] Antoine Girard. Dynamic triggering mechanisms for event-triggered control. *IEEE Transactions on Automatic Control*, 60(7):1992–1997, 2015.

[5] Dinh Cong Huong. Discrete-time dynamic event-triggered H_∞ control of uncertain neural networks subject to time delays and disturbances. *Optimal Control, Applications and Methods*, page doi.org/10.1002/oca.2945, 2022.

[6] Dinh Cong Huong. On event-triggered robust observer-based control problem of one-sided Lipschitz time-delay systems. *Asian Journal of Control*, 24:2234– 2243, 2022.

[7] Hassan K. Khalil. *Nonlinear Systems*. Prentice-Hall, 3rd edition, 2000.

[8] Winfried Lohmiller. *Contraction analysis of nonlinear systems*. PhD thesis, Massachusetts Institute of Technology, 1999.

[9] Winfried Lohmiller and Jean-Jacques E. Slotine. Contraction analysis: a practical approach to nonlinear control applications. In *IEEE International Conference on Control Applications*, pages 1–5, Trieste, Italy, 1998. IEEE.

[10] Winfried Lohmiller and Jean-Jacques E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998.

[11] David Lovelock and Hanno Rund. *Tensors, differential forms, and variational principles*. Dover, 1989.

[12] Nicolas Marchand, Sylvain Durand, and Jose Fermi Guerrero Castellanos. A general formula for event-based stabilization of nonlinear systems. *IEEE Tranactions on Automatic Control*, 58(5):1332–1337, 2013.

[13] Frédéric Mazenc, Michael Malisoff, and Corina Barbalata. Event-triggered control for continuous-time linear systems with a delay in the input. *Systems and Control Letters*, 159:105075, 2022.

[14] Frédéric Mazenc, Michael Malisoff, Corina Barbalata, and Zhong-Ping Jiang. Event-triggered control using a positive systems approach. *European Journal of Control*, 62:63–68, 2021.

[15] Nacim Meslem and Christophe Prieur. Event-based controller synthesis by bounding methods. *European Journal of Control*, 26:12–21, 2015.

[16] Romain Postoyan, Paulo Tabuada, Dragan Nesic, and Adolfo Anta. A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 60(4):982–996, 2015.

[17] Alexandre Seuret, Christophe Prieur, and Nicolas Marchand. Stability of non-linear systems by means of event-triggered sampling algorithms. *IMA Journal of Mathematical Control and Information*, 31(3):415–433, 2014.

[18] Eduardo D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, volume 6 of *Texts in Applied Mathematics*. Springer, 2nd edition, 1998.

[19] Paulo Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685, 2007.

[20] Huanqing Wang, Song Ling, Peter Xiaoping Liu, and Yuan-Xin Li. Control of high-order nonlinear systems under error-to-actuator based event-triggered framework. *International journal of control*, 95:2758–2770, 2022.

[21] Xiaofeng Wang and Michael Lemmon. On event design in event-triggered feedback systems. In *47th IEEE Conference on Decision and Control*, pages 2105–2110, Cancun, Mexico, 2008. IEEE.

[22] Lantao Xing, Changyun Wen, Zhitao Liu, Hongye Su, and Jianping Cai. Event-triggered adaptive control for a class of uncertain nonlinear systems. *IEEE Transactions on Automatic Control*, 62(4):2071–2076, April 2017.

[23] Taixiang Zhang, Xiaodi Li, and Shiji Song. Finite-time stabilization of switched systems under mode-dependent event-triggered impulsive control. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 52(9):5434–5442, 2021.

[24] Rui Zhao, Zhiqiang Zuo, and Yijing Wang. Event-triggered control for networked switched systems with quantization. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 52(10):6120–6128, 2022.

[25] Fairouz Zobiri, Nacim Meslem, and Brigitte Bidégaray-Fesquet. Event-triggered stabilizing controllers based on an exponentially decreasing threshold. In *3rd International Conference on Event Based Control, Communication and Signal Processing (EBCCSP 2017)*, Funchal, Portugal, May 2017. IEEE.

[26] Fairouz Zobiri, Nacim Meslem, and Brigitte Bidégaray-Fesquet. Event-triggered stabilizing controllers for switched linear systems. *Nonlinear Analysis: Hybrid Systems*, 36:100831, 2020.