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ABSTRACT

This article deals with the Cauchy problem for a nonlocal Zakharov equation. We will first recall the physical motivation (due to V.E. Zakharov) of this system. Then we will study the local Cauchy problem for certain initial data, and will identify the limit of the solutions when the ion velocity tends to infinity.

1 Origin of the nonlocal Zakharov system

The Physical theory which follows has been developped by V.E. Zakharov[5,6] to describe the Langmuir Oscillations in Plasma Physics.
We assume that:

1. the plasma is sufficiently uniform,
2. the magnetic field is sufficiently weak,
3. the nonlinearity level is not too high,
4. there are no transverse high frequency electromagnetic waves.

We consider the following equations that modelise the phenomena:
Linearised hydrodynamical equations

$$\frac{\partial}{\partial t}\delta n_e + \mathrm{div}(n_0+\delta n)\vec{V}_e = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\delta\vec{V}_e + \frac{3V_{T_e}^2}{n}\vec{\nabla}\delta n = \frac{e}{m_e}\vec{E}. \tag{2}$$

Maxwell's equation

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \mathrm{curlcurl}\vec{E} + \frac{4\pi e}{c^2}(n_0+\delta n)\frac{\partial}{\partial t}\delta\vec{V}_e = 0. \tag{3}$$

Vlasov's equation

$$\frac{\partial f_i}{\partial t} + (V.\vec{\nabla})f_i - n\frac{e}{m_i}\vec{\nabla}\varphi_{el}\frac{\partial f_i}{\partial V} = 0, \tag{4}$$

$$\delta n_i = \frac{en_0}{T_e}(\varphi_{el}-\varphi). \tag{5}$$

We moreover set

$$\vec{E} = \frac{1}{2}\left(\vec{\tilde{E}}\exp(-i\omega_{pl}t) + \vec{\tilde{E}}^*\exp(i\omega_{pl}t)\right), \tag{6}$$

and

$$\vec{\tilde{E}} = \vec{\nabla}\psi. \tag{7}$$

Eq. 1-7 imply that

$$\Delta(2i\omega_{pl}\psi_t + 3V_{T_e}^2\Delta\psi) = \omega_{pl}^2\mathrm{div}(\frac{\delta n}{n_0}\nabla\psi). \tag{8}$$

Two different hypothesis can be made:
First hypothesis
The nonlinear phenomena have such a long period that the ions have enough time to reach the Boltzmann distribution law in a low frequency field:

$$\frac{\delta n}{n_0} = -e\frac{\varphi_{el}}{T_i}. \tag{9}$$

After some computations and a change of scale, Eq. 8 and 9 lead to

$$\Delta(i\psi_t + \Delta\psi) = \mathrm{div}(|\nabla\psi|^2\nabla\psi). \tag{10}$$

This equation has been widely studied by T. Colin[1,2].
Second hypothesis
The ions do not have the time to reach this distribution:

$$\left(\frac{\partial^2}{\partial t^2} + 2\gamma_s\frac{\partial}{\partial t} - c_s^2\Delta\right)\delta n = \frac{1}{16\pi m_i}\Delta|E|^2. \tag{11}$$

We will suppose that the damping rate is zero, i.e. we neglect $2\gamma_s\frac{\partial}{\partial t}$.
After a change of scale, Eq. 11 and 8 become

$$\left\{\begin{array}{l}\Delta(i\psi_t+\Delta\psi)=\operatorname{div}(n\nabla\psi),\\ \frac{1}{c^2}n_{tt}-\Delta n=\Delta(|\nabla\psi|^2).\end{array}\right. \tag{12}$$

In what follows, we will not give any complete proof. They may be found in an article which is to appear[7].

2 Existence and uniqueness for the Cauchy Problem

We first consider the system 12 with $c=1$:

$$\left\{\begin{array}{l}\Delta(i\psi_t+\Delta\psi)=\operatorname{div}(n\nabla\psi),\\ n_{tt}-\Delta n=\Delta(|\nabla\psi|^2).\end{array}\right. \tag{13}$$

We set $B=\nabla(-\Delta)^{-1}\nabla.$ and $\varphi=\nabla\psi$. Eq. 13 is equivalent to

$$\left\{\begin{array}{l}i\varphi_t+\Delta\varphi=-B(n\varphi),\\ n_{tt}-\Delta n=\Delta(|\varphi|^2).\end{array}\right. \tag{14}$$

B is homogeneous of order 0 in the Fourier variables. Thanks to Calderón-Zygmund's theorem we have the following result:
For all $1<p<\infty$, there exists $C_{s,p}$ such that

$$\|Bf\|_{W^{s,p}}\leq C_{s,p}\,\|f\|_{W^{s,p}}. \tag{15}$$

2.1 *Theorem for the Cauchy Problem*

Theorem 1 *Let us consider the problem on R^N, $N=1,2,3$.*

$$\left\{\begin{array}{l}i\dot\varphi+\Delta\varphi=\nabla\Delta^{-1}\nabla.(n\varphi),\\ \ddot n-\Delta n=\Delta|\varphi|^2,\\ n(x,0)=n_0(x),\\ \partial_t n(x,0)=n_1(x),\\ \varphi(x,0)=\varphi_0(x),\end{array}\right. \tag{16}$$

with $n_0\in H^1, n_1\in L^2$ and $\varphi_0\in H^2$.
Then there exists a time $T>0$ depending only on $\|n_0\|_{H^1}$, $\|n_1\|_{L^2}$, $\|\varphi_0\|_{H^2}$ and N and a unique solution $(\varphi(t),n(t))$ to Eq. 16 which satisfies

$$\left\{\begin{array}{lcl}\varphi(t)&\in&\mathcal{C}^0([0,T];H^2)\cap\mathcal{C}^1([0,T];L^2),\\ \varphi(t)&\in&W^{1,8/N}(0,T;L^4),\\ n(t)&\in&\mathcal{C}^0([0,T];H^1)\cap\mathcal{C}^1([0,T];L^2)\cap\mathcal{C}^2([0,T];H^{-1}).\end{array}\right. \tag{17}$$

Thanks to Eq. 15, we have been inspired for the idea of the proof by the work of Ozawa and Tsutsumi[3]. We first make a change of variables (in order not to loose regularity at each step). We then use a fixed point method to find existence and uniqueness in a subset of the original functional space. To conclude, we prove uniqueness in the whole space and return to the initial variables.

2.2 *Setting of the fixed point method*

Setting $F = \partial_t \varphi$, we formally get

$$\begin{cases} iF_t + \Delta F = -B(\partial_t n(\varphi_0 + \int_0^t F ds) + nF), \\ n_{tt} - \Delta n = \Delta(|\varphi|^2), \\ \varphi = (-\Delta + 1)^{-1}\{iF + B(n(\varphi_0 + \int_0^t F ds)) + (\varphi_0 + \int_0^t F ds)\}, \end{cases} \tag{18}$$

with $F(0) = F_0$, $n(0) = n_0$, $\partial_t n(0) = n_1$.
We work in R^N with $1 \leq N \leq 3$. We use the following functional space:

$$X = [L^\infty(I; L^2) \cap L^{8/N}(I; L^4)] \oplus [L^\infty(I; H^1) \cap W^{1,\infty}(I; L^2)], \tag{19}$$

where $I = [0, T]$.
We set $N = (N_1, N_2)$ with

$$\begin{cases} N_1[F, n](t) = U(t)F_0 + i \int_0^t U(t-s)\{B(\partial_s n(\varphi_0 + \int_0^s F d\tau) + nF)\} ds, \\ N_2[F, n](t) = \cos(\omega t) n_0 + \omega^{-1} \sin(\omega t) n_1 + \int_0^t \omega^{-1} \sin(\omega(t-s)) \Delta |\varphi(s)|^2 ds, \\ (-\Delta + 1)\varphi = \{iF + B(n(\varphi_0 + \int_0^t F ds)) + (\varphi_0 + \int_0^t F ds)\}, \end{cases} \tag{20}$$

where $U(t)$ is the group generated by the Schrödinger operator and ω denotes $\sqrt{-\Delta}$ (multiplication by $|\xi|$ in Fourier variables).
We set

$$a = \max\{\|\varphi_0\|_{L^2}, \|\varphi_0\|_{L^4}, \|\Delta\varphi_0 + B(n_0\varphi_0)\|_{L^2}, \|n_0\|_{H^1} + \|n_1\|_{L^2}\}, \tag{21}$$

$$\begin{aligned} Y = \{(F(t), n(t)) \in X / \quad & \|F\|_{L^\infty(I;L^2)} \leq 2a, \|F\|_{L^{8/N}(I;L^4)} \leq 2\delta a, \\ & \|n\|_{L^\infty(I;H^1)} \leq 2, \left\|\frac{dn}{dt}\right\|_{L^\infty(I;L^2)} \leq 2a\}. \end{aligned} \tag{22}$$

The fixed point method will consist in solving

$$\begin{cases} N1[F, n] = F, \\ N2[F, n] = n. \end{cases} \tag{23}$$

$N : Y \to Y$ is a contraction if T is sufficiently small, thus we may find a fixed point.

2.3 *Return to the initial problem*

The fixed point method yields:

$$\begin{cases} F(t) & \in & \left[\bigcap_{j=0}^{1} \mathcal{C}^j([0,T];H^{-2j})\right] \bigcap L^{8/N}(0,T;L^4), \\ n(t) & \in & \bigcap_{j=0}^{2} \mathcal{C}^j([0,T];H^{1-j}), \\ \varphi(t) & \in & \mathcal{C}([0,T];H^2). \end{cases} \tag{24}$$

This regularity enables us to return to the initial variables; we get the desired regularity and two conservation laws:

$$\int |\varphi(t)|^2 = \int |\varphi_0|^2, \tag{25}$$

$$\int (|\nabla\varphi(t)|^2 + n(t)|\varphi(t)|^2 + \frac{1}{2}(\nabla\Phi(t))^2 + \frac{1}{2}n^2(t)) = \int (|\nabla\varphi_0|^2 + n_0|\varphi_0|^2 + \frac{1}{2}(\nabla\Phi_0)^2 + \frac{1}{2}n_0^2), \tag{26}$$

where $-\Delta\Phi = \dfrac{\partial n}{\partial t}$.

3 Limit as c tends to ∞

We now consider the problem

$$\begin{cases} \frac{1}{c^2} n_{tt} - \Delta(n + |E|^2) = 0, \\ iE_t + \Delta E + B(nE) = 0. \end{cases} \tag{27}$$

The result obtained in Section 2 still holds but T depends on c.
We shall work with variables in H^s with $s > \left[\dfrac{k}{2}\right] + 3$.
There is no need to prove uniqueness since this has been already done in some larger space.
The formal limit of the solution is the couple $(-|\tilde{E}|^2, \tilde{E})$ solution of

$$i\tilde{E}_t + \Delta\tilde{E} - B(|\tilde{E}|^2\tilde{E}) = 0. \tag{28}$$

3.1 *The Theorem for the Limit*

Theorem 2 *When c tends to ∞,*
$n^c + |E^c|^2 \to 0$ *in* $\mathcal{C}^0([0,T] \times R^k)$,
$\nabla(n^c + |E^c|^2) \to 0$ *in* $\mathcal{C}^0([0,T];H^{s-2})$,
$E^c \to \tilde{E}$ *in* $\mathcal{C}^1([0,T] \times R^k) \cap \mathcal{C}([0,T];\mathcal{C}^2)$,
where $\tilde{E}$ is the unique solution of

$$i\tilde{E}_t + \Delta\tilde{E} - B(|\tilde{E}|^2\tilde{E}) = 0. \tag{29}$$

This time we adapt the method of Schochet and Weinstein[4]. We first carry out a transformation of the system into a dispersive perturbation of a symmetric hyperbolic one. Then we prove the existence of a regular solution for a time independent of c and pass to the limit when c tends to ∞.

3.2 *Transformation of the system*

We set

$$\begin{cases} V = -\frac{1}{c}\Delta^{-1}\nabla n_t, \\ Q = n + |E|^2, \end{cases} \tag{30}$$

$$\sqrt{2}E = F + iG \text{ et } \sqrt{2}\nabla E = H + iL, \tag{31}$$

and

$$U =^t (Q, V, F, G, H, L), \tag{32}$$

then the system may be rewritten as

$$U_t + \sum_{j=1}^{k}\{R(A^j(U)U_{x_j}) + cC^jU_{x_j}\} + S(\tilde{B}(U)U) = K\Delta U. \tag{33}$$

R and S are Calderón-Zygmund's operators.
$\tilde{B}$ is a nonlocal operator.
K is an antisymmetric matrix.
C^j and $A^j(U)$ are symmetric matrices.
These properties will be very useful to make estimation in Sobolev spaces and to use the classical theory of hyperbolic equations.

3.3 *Existence of a regular solution for a time independent of c*

To this end, we use the following iteration scheme:

$$U^0(x,t) = U_0(x), \tag{34}$$

$$\frac{\partial U^{p+1}}{\partial t} + \sum_{j=1}^{k}\{R(A^j(U^p)U^{p+1}_{x_j}) + cC^jU^{p+1}_{x_j}\} + S(\tilde{B}(U^p)U^{p+1}) = K\Delta U^{p+1}, \tag{35}$$

$$U^{p+1}(x,0) = U_0(x). \tag{36}$$

One can show the following estimates.

- $\forall p \geq 0 \qquad |||U^p|||_{s,T} \leq \delta,$
- $\forall p \geq 0 \qquad |||U^{p+1} - U^p|||_{0,T} \leq C|||U^p - U^{p-1}|||_{0,T},$
  with $C < 1$.

- Then
$$\begin{array}{l} U^p \to U \text{ in } L^\infty(0,T;L^2), \\ U^p \text{ is bounded in } L^\infty(0,T;H^s). \end{array} \tag{37}$$

  Then $U \in \mathcal{C}([0,T];\mathcal{C}^1)$ and the solution is a classical one.
  Moreover we have
$$\begin{array}{l} U \in Lip([0,T];H^{s-2}), \\ U \in \mathcal{C}([0,T];H^s) \cap \mathcal{C}^1([0,T];H^{s-2}). \end{array} \tag{38}$$

The time T is independent of c (because $\mathcal{C}^j$ is symmetric).
To gain some regularity we use the theory of commutators.
There is no problem in returning to the initial variables and we get

Theorem 3 *Let $s \geq \left[\frac{k}{2}\right] + 3$. For $n_0 \in H^s$, $n_1 \in H^{s-1}$ and $E_0 \in H^{s+1}$, there exists a unique solution to system 27 endowed with the following initial data:*

$$\left\{ \begin{array}{r} n(0,x) = n_0(x), \\ \partial_t n(0,x) = n_1(x), \\ E(0,x) = E_0(x), \end{array} \right. \tag{39}$$

on a time interval $[0,T]$, T not depending on c but only on $\|n_0\|_{H^s}$, $\|n_1\|_{H^{s-1}}$ and $\|E_0\|_{H^{s+1}}$.
Moreover, for all $t \in [0,T]$ we have the estimate

$$\|E^c\|_{H^{s+1}} + \|E^c_t\|_{H^{s-1}} + \|n^c\|_{H^s} + \frac{1}{c}\|n^c_t\|_{H^{s-1}} + \frac{1}{c^2}\|n^c_{tt}\|_{H^{s-2}} \leq \ Cst \tag{40}$$

Thanks to the two conservation laws Eq. 25 and 26 we can, as for the classical Zakharov equation ($B = -I$), show that the solutions are global in time in the 1-dimensional case and also in the 2-dimensional case when the initial data are sufficiently small.

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1. T. Colin, Sur une équation de Schrödinger non linéaire et non locale intervenant en physique des plasmas. *C.R.A.S. Paris*, **314**, série I, (1992) 449-453.

2. T. Colin, On a nonlocal, nonlinear Schrödinger equation occuring in plasma Physics. *Publication CMLA*, ENS Cachan, **9208** (1992).

3. T. Ozawa, Y. Tsutsumi, Existence and Smoothing Effect of Solutions for the Zakharov Equations. 1990.

4. S.H. Schochet, M.I. Weinstein, The Nonlinear Schrödinger Limit of the Zakharov Equations Governing Langmuir Turbulence. *Commun. Math. Phys.*, **106**, (1986) 569-580.

5. V.E. Zakharov, Collapse and Self-Focusing of Langmuir Waves. in *Basic Plasma Physics II*, ed. A.A. Galeev and R.N. Sudan, (North Holland, 1984).

6. V.E. Zakharov, Collapse of Langmuir Waves. *Soviet Physics JETP*, **Vol. 35, 5** (1972) 908-914.

7. B. Bidégaray, On a nonlocal Zakharov equation, to appear