A NON LOCAL SCHRÖDINGER MODEL FOR THE PROPAGATION OF WAVES IN A PHOTOREFRACTIVE MEDIUM

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Abstract

We present various mathematical results (Cauchy problem, solitary waves) for the Zozulya–Anderson model which describes the propagation of an optical wave through a photorefractive medium. This is a joint work with Jean-Claude Saut.

1 Introduction

Photorefractive media are optical material, highly anisotropic, with memory effects. They are commonly used to realize optically induced gratings or for holographic applications.

The propagation of an optical wave in insulating or semi-insulating electrooptical crystals induces a charge transfer. The new distribution of charges ρ induces in turn an electric field $\mathbf{E}$, with $\nabla \cdot (\widehat{\varepsilon}\mathbf{E}) = \rho$. This field derives from a potential φ and produces a variation δn of the refraction index in the main direction of the photovoltaic effect (which we choose here to be x): $\delta n \propto \partial_x \varphi$. The main characteristics of this effect are the following: 1- Sensibility to energy (and not to the electric field). This recalls the Kerr effect and the cubic nonlinear Schrödinger equation. 2- Nonlocal effect (charge distributions and the electric field are not located at the same position). 3- Inertia (charges need a certain time to move). We will not take this into account here. 4- Memory and reversibility (in the dark the space charge, and therefore the index variation, is persistent but an uniform light redistributes uniformly all charges — this yields applications to holography). We will also neglect this effect here, reducing our study to material where only electrons are moving.

2 Mathematical setting

A complete mesoscopic model for the modeling of photo-refractive media is the Kukhtarev model [6]. In the case when the charges that contribute to the photorefractive effect are only electrons (insulating media), an asymptotic study allows to derive a macroscopic model, the Zozulya–Anderson model. The complete assumptions and approximations made are precisely described in [2].

The description of the propagation of a laser through the photorefractive material is given by a Schrödinger equation using paraxial and envelope approximations. The propagation axis is chosen to be z and all constants are taken to be 1, which can be justified rigorously using dimensionless variables (see [2]). One obtains

$$\left[\partial_z - \frac{i}{2}\nabla_\perp^2\right] A = -iA\partial_x\varphi.$$

If we specify a material (e.g. $LiNbO_3$) and therefore symmetries of the tensor $\widehat{\varepsilon}$, we can write an equation for φ which reads

$$\nabla_\perp^2\varphi + \nabla_\perp \ln(1+|A|^2)\cdot\nabla_\perp\varphi = \partial_x \ln(1+|A|^2).$$

These are the Zozulya–Anderson equations [10].

If we look at a wider class of materials we may have different signs for the nonlinearity (in reference to the cubic nonlinear Schrödinger equation, the case $a = 1$ is classically called the focusing case, and $a = -1$ the defocusing case). Besides mathematicians are more accustomed to use t as the evolution variable. Finally logarithms are difficult to handle in the mathematical analysis (although natural if we look at 1D solitary waves, see below), we therefore rewrite also the equation for φ. We finally impose an initial data A_0 in some convenient (see below) functional space and obtain the system

$$\text{(ZA)}\quad \left\{\begin{array}{rcl} i\partial_t A + \Delta A &=& -aA\partial_x\varphi, \\ \operatorname{div}\left((1+|A|^2)\nabla\varphi\right) &=& \partial_x(|A|^2), \\ A(\cdot,0) &=& A_0. \end{array}\right.$$

The main effects take place in the t (propagation) and the x directions. It is therefore natural to study the equations with no dependence in the y variable. In the one dimensional case, since we assume furthermore that no external field is applied, we can immediately infer that $\partial_x\varphi = |A|^2/(1+|A|^2)$. We therefore consider the saturated non linear Schrödinger equation

$$\text{(SNLS)}\quad \left\{\begin{array}{rcl} i\partial_t A + \Delta A &=& -a\dfrac{|A|^2 A}{1+|A|^2}, \\ A(\mathbf{x},0) &=& A_0(\mathbf{x}), \end{array}\right.$$

where $a = \pm 1$, $A = A(\mathbf{x},t)$ and $\mathbf{x} \in \mathbb{R}^d$. We have derived this equation for $d = 1$, but give here results for a general d, which also arises in other contexts, such as the propagation of a laser beam in gas vapors [9].

In the two-dimensional case (ZA) can be viewed as a saturated version of a Davey–Stewartson system. Namely, replacing $1+|A|^2$ by 1 in the left hand-side of (ZA) we obtain a Davey–Stewartson system of the elliptic–elliptic type (see Ghidaglia and Saut [4]).

3 The Cauchy problem

3.1 The generalized saturated NLS equation

Theorem 1 *(i) Let $A_0 \in L^2(\mathbb{R}^d)$. Then there exists a unique solution $A \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$ of (SNLS) which satisfies furthermore $\mathcal{E}(t) = \mathcal{E}(0)$ for all $t \in \mathbb{R}$, where*

$$\mathcal{E}(t) \equiv \int_{\mathbb{R}^d} |A(t)|^2 d\mathbf{x}.$$

(ii) Let $A_0 \in H^1(\mathbb{R}^d)$. Then the solution above satisfies $A \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ and $\mathcal{H}(t) = \mathcal{H}(0)$ for all $t \in \mathbb{R}$, where

$$\mathcal{H}(t) \equiv \int_{\mathbb{R}^d} \left[|\nabla A(t)|^2 d\mathbf{x} + a \ln(1+|A(t)|^2) \right] d\mathbf{x}.$$

The proof follows the usual steps for nonlinear Schrödinger equations. Contrarily to the context of the usual nonlinear cubic Schrödinger equation, the solution is global in time, whatever the sign of a. Saturation prevents from blowing up.

We would like to mimic this proof to treat (ZA). To this aim we would like to express A in terms of φ for say $A \in L^2(\mathbb{R}^2)$. With such a data A, we indeed have a unique φ in some convenient space but no Lipschitz regularity for the mapping $A \mapsto \varphi$, which is required to perform some fixed point procedure. To ensure this we will have to assume $A \in H^2(\mathbb{R}^2)$.

Theorem 2 *Let $A_0 \in H^2(\mathbb{R}^2)$. Then there exists $T_0 > 0$ and a unique solution $(A, \nabla\varphi)$ of (ZA) such that $A \in \mathcal{C}([0,T_0]; H^2(\mathbb{R}^2))$ and $\nabla\varphi \in \mathcal{C}([0,T_0]; H^2(\mathbb{R}^2))$. Moreover for all $0 \le t \le T_0$*

$$\|A(t)\|_{L^2(\mathbb{R}^2)} = \|A_0\|_{L^2(\mathbb{R}^2)}$$

and

$$\int_{\mathbb{R}^2} (1 + \frac{1}{2}(t)|A|^2)|\nabla\varphi(t)|^2 d\mathbf{x} \le \frac{1}{2} \int_{\mathbb{R}^2} |A_0|^2 d\mathbf{x}.$$

The proof of this result necessitates many steps. Uniqueness results follows from simple energy estimates. Then we derive a H^2 *a priori* estimate of the solution of (ZA). We introduced an approximate system for which the well-posedness stems from classical fixed point arguments and the convergence towards (ZA) is first obtained in $L^\infty(0,T; H^2(\mathbb{R}^2))$ using the Aubin–Lions compactness lemma [7]. The final existence result in $\mathcal{C}([0,T]; H^2(\mathbb{R}^2))$ follows from the Bona–Smith approximation [3]. We do not have any hint on whether this local solution is global or not.

4 Solitary waves

4.1 First integrals for 1D solitary waves

1D bright solitary waves are sought for in the form $A(x,t) = e^{i\omega t}u(x)$ (see [8]), where A is a solution to (SNLS). The function u is supposed to have a maximum at $x = 0$ ($u(0) = u_m > 0$ and $u'(0) = 0$). We furthermore want that for $x \to \infty$, $u(x) \to 0$ and $u'(x) \to 0$. This yields a unique possible frequency for the solitary wave, namely

$$\omega = a\left(1 - \frac{\ln(1+u_m^2)}{u_m^2}\right)$$

and imposes $a = 1$ (focusing case). The bright solitary wave is solution to the first order equation

$$u'(x) = -\mathrm{sign}(x)\sqrt{\ln(1+u^2) - \frac{u^2}{u_m^2}\ln(1+u_m^2)}.$$

4.2 Non existence of solitary waves for (SNLS) and (ZA)

Consider now the (bright) solitary wave solutions of (SNLS) of the type $A(\mathbf{x},t) = e^{i\omega t}U(\mathbf{x})$, where $U \in H^1(\mathbb{R}^d)$. It is solution to the elliptic equation

$$-\Delta U + \omega U = a\frac{|U|^2 U}{1+|U|^2}, \qquad U \in H^1(\mathbb{R}^d).$$

Proposition 3 *No non-trivial ($U \not\equiv 0$) solitary wave of (SNLS) exists when*
(i) $a = -1$ (defocusing case), for $\omega \ge 0$. (ii) $a = 1$ (focusing case) and $\omega \ge 1$. (iii) $a = \pm 1$ if $\omega < 0$ provided $|U|^2/(1+|U|^2) = O(1/|\mathbf{x}|^{1+\varepsilon})$, $\varepsilon > 0$ as $|\mathbf{x}| \to +\infty$.

We now look for solitary wave solutions of (ZA), that is solutions of the form $(e^{i\omega t}U(x), \phi(x))$ with $x \in \mathbb{R}^d$, $\omega \in \mathbb{R}$, $U \in H^1(\mathbb{R}^d)$ and $\phi \in H$. Thus (U,ϕ) should satisfy the system

$$\text{(RSW)} \left\{ \begin{array}{rcl} -\Delta U + \omega U &=& aU\partial_x\phi, \\ \mathrm{div}((1+|U|^2)\nabla\phi) &=& \partial_x(|U|^2). \end{array} \right.$$

The existence of non-trivial solutions of (SW) is an open problem. Note that (SW) does not seem to be the Euler–Lagrange equation associated to a variational problem. We have however:

Proposition 4 *(i) Let $a = -1$ (defocusing case). Then no non-trivial solution of (SW) exists for $\omega \geq 0$.*
(ii) Let $a = 1$ (focusing case). No non-trivial solution of (SW) exists for $\omega \geq 1$.
(iii) Let $a = \pm 1$. No non-trivial solution of (SW) exists if $\omega < 0$ provided $\partial_x \phi = O(1/|\mathbf{x}|^{1+\varepsilon})$, $\varepsilon > 0$ as $|\mathbf{x}| \to +\infty$.

In both propositions, (i) and (ii) follow from simple energy estimates and (iii) from the classical result of Kato [5] on the absence of embedded eigenvalues.

4.3 Existence of solitary waves for (SNLS)

We now turn to the existence of non-trivial H^2 solutions of

$$-\Delta U + \omega U = \frac{|U|^2 U}{1 + |U|^2}$$

when $0 < \omega < 1$. We will look for real radial solutions $U(\mathbf{x}) = u(|\mathbf{x}|) \equiv u(r)$ and thus consider the ODE problem

$$\text{(RSW)} \begin{cases} -u'' - \dfrac{d-1}{r} u' + \omega u = \dfrac{u^3}{1+u^2}, \\ u \in H^2(]0, \infty[), \qquad u'(0) = 0. \end{cases}$$

Theorem 5 *If $a = 1$ and $0 < \omega < 1$, there exists a non-trivial positive solution of (RSW).*

This a consequence of a classical result of Berestycki, Lions and Peletier [1].

5 Conclusion

We have derived from the Kukhtarev equations an asymptotic model for the propagation of light in a photorefractive medium.

The 1D asymptotic model is a saturated nonlinear Schrödinger equation the Cauchy problem of which is studied (in any space dimension) in L^2 and H^1. We also prove the existence of solitary waves in 1 and higher dimensions. An interesting and open issue would be to study the transverse stability of the 1D solitary waves in the framework of the asymptotic model.

For the 2D asymptotic model (the Zozulya–Anderson model) we also have studied the Cauchy problem and the non-existence of solitary waves. The question of imposing other boundary conditions, not vanishing in one space direction, should also be addressed to treat a wider range of experimental applications. We are already able to find first integrals of (dark) solitary waves in this context.

Memory effects also certainly lead to interesting equations from the mathematical point of view. This necessitates however a new full derivation, which is a difficult and tedious task.

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