Second order corrections to the finite volume upwind scheme for the 2D Maxwell equations

B. Bidégaray
MIP - CNRS UMR 5640
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse Cedex
France

J.-M. Ghidaglia
CMLA - CNRS UMR 8635
ENS de Cachan
61 avenue du président Wilson
94235 Cachan Cedex
France

ABSTRACT *When computing solutions to Maxwell equations with finite volumes methods one often faces mesh dependent structures. We describe a way to add second order corrections to the finite volume upwind scheme for the 2D Maxwell equations that are designed to overcome this difficulty. This is done by using exact solutions to the wave equation for each component of the electromagnetic field. We illustrate the method with numerical results on simple test cases.*

Key Words : 2D Maxwell equations, finite volumes, upwind scheme.

1. Introduction

When computing solutions to Maxwell equations with finite volumes methods one often faces mesh dependent structures. Indeed, fluxes are computed across edges that may have privileged directions in some parts of the computational domain. Our goal is to write a modification of the classical upwind finite volume method that handles this drawback.

Contribution to "Finite Volumes for Complex Applications II – Problems and Perspectives, Duisburg, Germany", eds. R. Vilsmeier, F. Benkhaldoun, D. Hänel, pp. 483-490, Hermes, 1999.

Here we more precisely address the 2D Maxwell equations in the TE_z polarization. Setting $\mathbf{Q} = (Q_1, Q_2, Q_3) = (E_x, E_y, H_z)$ they read

$$\left\{ \begin{array}{lcl} \varepsilon \dfrac{\partial}{\partial t} Q_1 & = & \dfrac{\partial}{\partial y} Q_3, \\ \varepsilon \dfrac{\partial}{\partial t} Q_2 & = & -\dfrac{\partial}{\partial x} Q_3, \\ \mu \dfrac{\partial}{\partial t} Q_3 & = & -\dfrac{\partial}{\partial x} Q_2 + \dfrac{\partial}{\partial y} Q_1. \end{array} \right. \tag{1}$$

In an homogeneous domain (i.e. ε and μ constant) each component of $\mathbf{Q}$ satisfies the wave equation

$$\frac{\partial}{\partial t^2} u - c^2 \Delta u = 0 \tag{2}$$

where $c^2 \varepsilon \mu = 1$. Both formulations are used to derive our scheme.

2. Leading order terms

The computational domain is decomposed into triangles. Integrating (2) on one of them, K, and on the time interval $[0, t]$, one gets

$$\frac{d}{dt} \int_K u d\mathbf{x} = \frac{d}{dt} \int_K u d\mathbf{x} \bigg|_{t=0} + c^2 \int_{\partial K} \frac{\partial}{\partial n} \left(\int_0^t u(\sigma, s) ds \right) d\sigma. \tag{3}$$

The computation of $I_K = \dfrac{d}{dt} \displaystyle\int_K u d\mathbf{x} \bigg|_{t=0}$ is performed through the usual upwind scheme (see e.g. [GHI 96]) for hyperbolic systems using formulation (1).

More precisely equation (1) is written as $\mathbf{Q}_t + \mathrm{div}(A\mathbf{Q}) = 0$ and I_K reads $I_K = -\displaystyle\int_{\partial K} A(\mathbf{n}) \mathbf{Q} d\sigma$ where $\mathbf{n}$ is the external unit normal vector on the boundary of the triangle K, ∂K. The eigenvalues of $A(\mathbf{n})$ are 0 and $\pm c$. For the computation of I_K this integral may be split in three contributions considering separately each edge of K. Then we decompose $A(\mathbf{n})\mathbf{Q}$ over the eigenvector basis of $A(\mathbf{n})$ and associate $\mathbf{Q} = \mathbf{Q}(K) =$ to eigenvalue c and $\mathbf{Q} = \mathbf{Q}(\tilde{K})$ to eigenvalue $-c$ where $\tilde{K}$ is K's neighbor across the considered edge of K. This is the usual upwind scheme. This is a particular case of our scheme and we will use it to evaluate its performances.

3. Second order corrections

The second part, $U_K = c^2 \int_{\partial K} \frac{\partial}{\partial n} \left(\int_0^t u(\sigma, s) ds \right) d\sigma$, which happens to be a second order correction, is computed using the exact solution of the wave equation (2) with initial condition 1 on triangle K and 0 elsewhere. This approach has already been used by Abgrall [ABG 94], Gilquin *et al* [GIL 91, GIL 94] and Chaïra [CHA 95] for solving the Riemann problem for gas dynamics and writing Euler equations as a wave equation, but in our case we have no conformal invariance.

For the computation of U_K we set $w(\mathbf{x}, t) = \int_0^t (u(\mathbf{x}, t) - u_0(K)) d\tau$. The insertion of u_0 which is formally bound to disappear in the next differentiation ensures that we preserve a constant solution over space (and time) if the initial data is constant, which is the first step towards flux conservation. We notice that this function w is solution to the wave equation

$$\frac{\partial}{\partial t^2} w - c^2 \Delta w = f \tag{4}$$

with initial data $w_0 \equiv 0$, initial time derivative $w_1 = u_0 - u_0(K)$ and right hand side $f = u_1 = -\mathrm{div}(A(u_0 - u_0(K))$. With this notation, $U_K = c^2 \int_{\partial K} \frac{\partial w}{\partial n}(\sigma, t) d\sigma$. An exact solution to this wave equation is given by Kirchoff formulae

$$w(x, y; t) = \frac{\partial}{\partial t} G(x, y; t) * w_0 + G(x, y; t) * w_1 + \int_0^t ds G(x, y; s) * f$$

where $G(x, y; t) = \frac{1}{2\pi c} \frac{H(c^2t^2 - x^2 - y^2)}{\sqrt{c^2t^2 - x^2 - y^2}}$ and H denotes the Heaviside function. Hence the computation of U_K leads to multiple integrals on the edges of the triangles that read (C and C' being equal or adjacent edges)

$$W_0 = - \int_{C \times C'} \frac{c}{2\pi} \frac{H(c^2t^2 - |\sigma - \sigma'|^2)}{\sqrt{c^2t^2 - |\sigma - \sigma'|^2}} d\sigma d\sigma'$$

and

$$W_1 = - \int_{C \times C'} \frac{ct}{2\pi} \frac{H(c^2t^2 - |\sigma - \sigma'|^2)(\sigma - \sigma') \cdot \mathbf{n}_C}{|\sigma - \sigma'|^2 \sqrt{c^2t^2 - |\sigma - \sigma'|^2}} d\sigma d\sigma'.$$

that may be expressed after tedious calculations by means of classical functions of the variables (lengths and angles in the triangle). We make an approximation here by computing only integrals over edges of K or its neighbors but not

triangles that only have one vertex in common with K, otherwise the computation of U_K would be exact.
This part of the derivation of our scheme is specific to the 2D case since the fundamental solution G to the wave equation has different expressions in different dimensions. The 1D case is easy to compute but is of no interest since there is no point in correcting direction dependent structures. After this modification the scheme is no longer a finite volume scheme : there is no exact balance of fluxes.

The formula for U_K shows some discontinuity at time $t = 0$. For $C' = C$, $W_0 = \dfrac{2c^2t - cl\pi}{2\pi}$ and $W_1 = 0$, and if we denote by α the angle between edges C and C' when are distinct,

	α acute	α obtuse
W_0	$-\dfrac{c^2t}{2\pi}\dfrac{\pi - \lvert\alpha\rvert}{\lvert\sin\alpha\rvert}$	$-\dfrac{c^2t}{2\pi}\dfrac{\lvert\pi - \alpha\rvert}{\lvert\sin\alpha\rvert}$
W_1	$\dfrac{ct}{8}\left(3\cos\alpha - \dfrac{2\cos\alpha + 1}{\lvert\sin\alpha\rvert}\right)$	$\dfrac{ct}{8}\left(\lvert\cos\alpha\rvert - \dfrac{1}{\lvert\sin\alpha\rvert}\right)$

These formulae are continuous for $\alpha = \frac{\pi}{2}$ and are a puzzle for computer algebra systems. To obtain explicit formulae and avoid numerical integration is a major advantage for numerical computations. This computation is performed with a condition on the size of triangles and the time step (that implies, in particular that no length appears in the final result). More general conditions may be taken into account but then the integration should be performed on more triangle edges. To take account of the discontinuity at time $t = 0$, we do not use the full correction but only a fraction of it given by a factor $\theta \in [0, 1]$. This means that we make a balance between time t and time 0 where $U_K = 0$. Since we deal with a second order correction this does not affect the consistence of the scheme, whatever the choice of θ is. The CFL ratio and this parameter θ are the two parameters to tune in order to obtain a "good" scheme. They are strongly linked in 1D but may be chosen independently in 2D. Numerical results show that $\theta = 0$ is not the best choice with respect to norm conservations for example. But the best θ also depends on the mesh and is therefore difficult to choose.

4. Numerical results

This work is still in progress and up to now only simple test cases have been performed. Different boundary conditions have been implemented (following [ENG 77, ENG 79, JOL 89], namely absorbing boundary conditions, perfect conducting surfaces and perfect reflecting boundaries. Different initial conditions and incident fields have also been tested.

The main drawback of the second order correction is that it induces some extra dissipation. This is very easy to see on a 1D equivalent of our correction. This dissipation is larger for a large θ. This leads us to choose $\theta \in [0, .3]$ for numerical simulations. The best conservation of norms is usually obtained in this interval. The following figure shows that for a test where the L^2 and L^∞ norm is supposed to be conserved, the choice the upwind scheme without correction ($\theta = 0$) is not the better scheme for norm conservation.

On Fig. 1 only the L^∞ norm of E_x is represented. The different curves correspond to different meshes for the same test case. Curves are similar for other components of the field or the L^2 norm. The fact that curves are decreasing is not generic but specific to this field in the computed case. Our correction seem to benefit finer meshes but criteria to adjust θ to a particular mesh are not obvious to find.

Since we only add a small perturbation to the original scheme we can not expect any real improvement of its main defaults, like phase shift. The initial purpose of the introduction of our U_K was the correction of mesh dependent structures and we have tested the propagation of a wave front with different angle of incidence. For small angles (otherwise our boundary conditions have to be improved) we show a real improvement of the straightness of the front. This shown on Fig. 2 and 3.

5. Perspectives

Perspectives of this work may be found in different directions. First we may go towards more realistic test cases and try to model real physical structures. A complete study of the case of an inhomogeneous media has to be performed. This would also contribute to more physical test. The 3D case is also of interest and formal calculations of the correction have to be derived in this context.

6. Bibliography

[ABG 94] Abgrall R., <<Approximation du problème de Riemann vraiment multidimensionnel des équations d'Euler par une méthode de type Roe (I) and (II)>>, *C.R. Acad. Sci. Paris 319*, 1994, p. 499-504 and 625-629.

[CHA 95] Chaïra S., Sur la résolution numérique des équations d'Euler de la dynamique des gaz par des schémas multidimensionnels, PhD Thesis, Ecole Normale Supérieure de Cachan, 1995.

[CIO 93] Cioni J.-P., Fezoui L. and Steve H., <<A parallel time-domain Maxwell solver using upwind schemes and triangular meshes>>, *Rapport INRIA 1867*, 1993.

[ENG 77] Engquist B. and Majda A., <<Absorbing Boundary Conditions for the Numerical Simulation of Waves>>, *Math. Comput, 31*, 1977, p. 629-651.

[ENG 79] Engquist B. and Majda A., <<Radiation Boundary Conditions for Acoustic and Elastic Wave Calculations>>, *Commun. Pure Appl. Math., 32*, 1979, p. 313-257.

[GHI 96] Ghidaglia J.-M., Kumbaro A. and Le Coq G., <<Une méthode 'volumes finis' à flux caractéristiques pour la résolution numérique des systèmes hyperboliques de lois de conservation>>, *C. R. Acad. Sci., série I, 332*, 1996, p. 981-988.

[GIL 91] Gilquin H. and Laurens J., <<Problèmes de Riemann multidimensionnels pour les systèmes hyperboliques linéaires>>, 1991.

[GIL 94] Gilquin H., Laurens J. and Rosier C., <<The explicit solution of the bidimensional Riemann problem for the linearized gaz dynamics equations>>, *ENS de Lyon, UMPA 133*, 1994.

[JOL 89] Joly P. and Mercier B., <<Une nouvelle condition transparente d'ordre 2 pour les équations de Maxwell en dimension 3>>, *Rapport INRIA, 1047*, 1989.

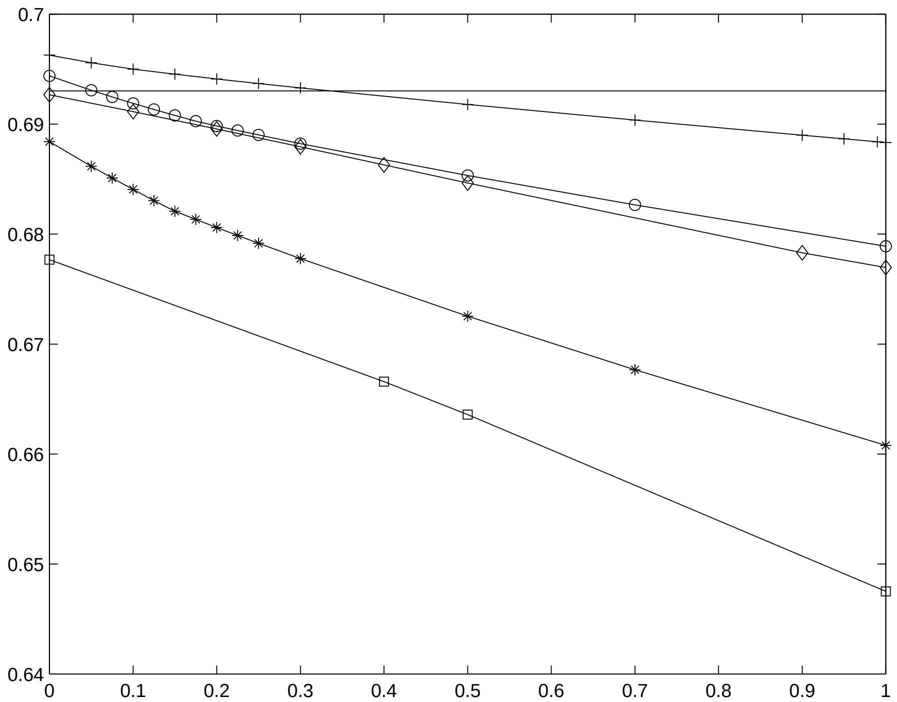

FIG. 1: *Norm conservation*
The L^∞ norm is represented as a function of θ and for different meshes, the exact L^2 norm being the horizontal line. The 'cross'-curve correspond to the finer irregular mesh (others are 'star' and 'circle'-curves) and the 'diamond'-curve is a regular mesh that is finer than the 'square'-curve.

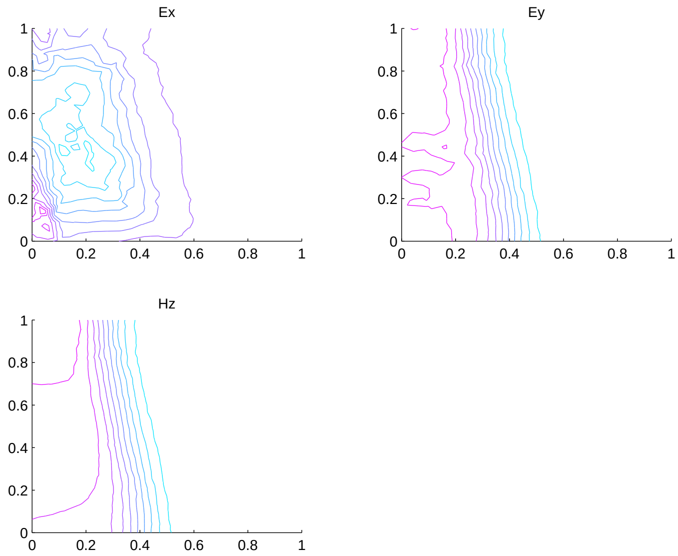

FIG. 2: *Front straightness* : $\theta = 0$

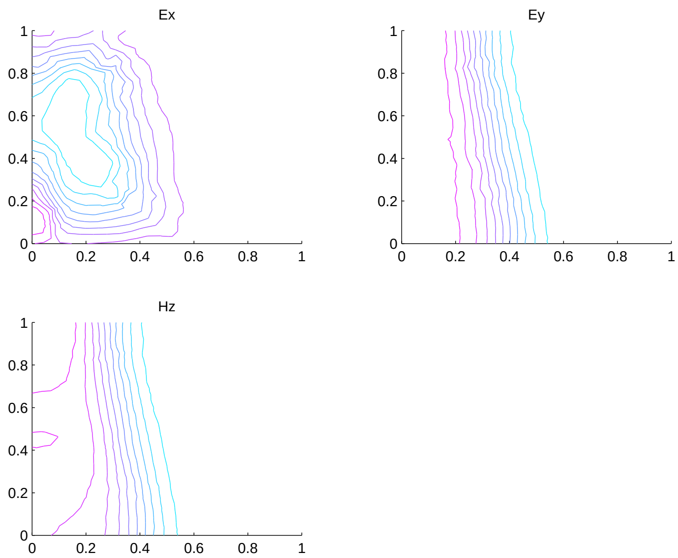

FIG. 3: *Front straightness* : $\theta = 0.1$