Level crossing sampling of strongly monoHölder functions

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***Abstract*—We address the problem of quantifying the number of samples that can be obtained through a level crossing sampling procedure for applications to mobile systems. We specially investigate the link between the smoothness properties of the signal and the number of samples, both from a theoretical and a numerical point of view.**

I. Introduction

An important issue in the design of mobile systems is to increase their autonomy and/or reduce their size and weight. This can be achieved by reducing their power consumption by processing signal with a smaller number of samples. For a large class of signals, especially sporadic signals, non-uniform sampling leads to a reduced number of samples, compared to a Nyquist sampling [7], [9], [10], [13]. A way to obtain such samples is to use specific system architectures (e.g. event-driven). These architectures take samples each time some specific event occurs, typically specific voltage levels are crossed. We can therefore design simple analog circuits, with low power consumption, to acquire information, possibly at high speed. Here we consider a system where amplitudes are selected thanks a M-bit asynchronous analog-to-digital converter (AADC) and 2^M levels are predefined in the voltage range.

In this article we want to understand on which signal characteristics the number of samples depend. An intuitive look at the problem indicates that the more the signal is oscillating locally the higher the number of samples is. The number of samples at the neighborhood of some point may then be related to the *local smoothness* of the signal, that is to its so–called *Hölder regularity*. To put in evidence this relationship, we consider toy models of signals whose smoothness properties are perfectly known at each point. We then perform numerical simulations and link the sample reduction rate with this regularity. The next step, which will be the purpose of a forthcoming paper, will then be to consider signals whose regularity may change from point to point such as multifractional or multifractal signals. We then intend to apply our results to biological signals such as EEG signals or fMRI data which are well-known to be both highly irregular and non stationary signals, and which provide interesting ranges of application for non-uniform signal processing.

II. Algorithm

In the event-driven systems, the signals are not sampled at totally arbitrary times. Indeed there are local clocks that measure the time elapsed since the previous sample was taken. Therefore we can consider that the samples are taken at some multiples of some basis time t_b. The mathematical algorithm that is used to mimic the AADC is the following:

- **Step 1:** generate uniform samples on $[0,1]$ with sampling interval t_b;
- **Step 2:** for each sample replace the amplitude by the value of the level just below;
- **Step 3:** decimate the samples so as to keep only one (the last) sample when consecutive samples have the same amplitude.

III. Mathematical interpretation

Up to some time re-scaling we suppose that the precision of the local clock that measures time delays is $t_b = 2^{-j}$, for $j \in \mathbb{N}$. At best we only know the function f by its samples at times $k2^{-j}$, $k \in \mathbb{Z}$. At scale 2^{-j}, we define the intervals

$$I_{j,k} = [k2^{-j}, (k+1)2^{-j}[.$$

A. Faber–Schauder hierarchical basis

We define the Faber–Schauder hierarchical basis as defined in [5]. Let V_j be the space of continuous functions, which are affine on intervals $I_{j,k}$, $k \in \mathbb{Z}$. We can uniquely define the linear interpolation f_j of f at scale 2^{-j} by $f_j(k2^{-j}) = f(k2^{-j})$, for all $k \in \mathbb{Z}$. Let $\varphi(x) = \max\{0, 1-|x|\}$. A natural basis for V_j is given by the functions $\varphi_{j,k} = \varphi(2^j \cdot - k)$, $k \in \mathbb{Z}$. In this basis, we have the unique representation

$$f_j = \sum_{k\in\mathbb{Z}} f(k2^{-j})\varphi_{j,k}.$$

B. Interpretation

We now suppose that f is compactly supported in $[0,1]$. In the previous notations we will only need $k = 0, \dots, 2^j - 1$. We assume that levels are uniformly spaced by some quantum 2^{-M}. The level crossing algorithm can be described as follows.

Step 1: approximation in V_j. We only know $f_j(k2^{-j}) = f(k2^{-j})$.

Step 2: level crossing. We denote $\lfloor x \rfloor$ the integer part of x, namely $\lfloor x \rfloor = \inf\{n \in \mathbb{N},\, x \leq n\}$. We then define

$$\tilde{f}_j = \sum_{k \in \mathbb{Z}} 2^{-M} \left\lfloor 2^M f(k2^{-j}) \right\rfloor \varphi_{j,k}.$$

Step 3: decimation of $\tilde{f}_j$. We only keep a subsequence of $k = 0, \ldots, 2^j - 1$, defined by induction: $k_0 = 0$ and

$$k_{i+1} = \min\{k \geq 1 + k_i / \left\lfloor 2^M f(k2^{-j}) \right\rfloor \neq \left\lfloor 2^M f(k_i 2^{-j}) \right\rfloor\}.$$

To be able to reconstruct $\tilde{f}_j$, we only store the couples $(\delta t_i, a_i)$ where $\delta t_i = (k_i - k_{i-1})2^{-j}$, $i \geq 1$ is the delay since the last sample, and $a_i = 2^{-M} \left\lfloor 2^M f(k_i 2^{-j}) \right\rfloor$ is the amplitude of the sample. There is no approximation in Step 3, we only do not store redundant data.

IV. Mathematical properties

We now want to illustrate through numerical experiments that the properties of our algorithm can be related to smoothness properties of the sampled signal.

A. MonoHölderian functions

Definition 1 (Hölder space $\mathcal{C}^\alpha$): Let $\alpha \in (0,1)$. The function f belongs to the Hölder space $\mathcal{C}^\alpha([0,1])$ if there exists a constant C such that for all $(x,y) \in [0,1]^2$,

$$|f(x) - f(y)| \leq C|x-y|^\alpha \,.$$

The following definition has been introduced in [12].

Definition 2 (Anti-Hölderian functions): Let $\alpha \in (0,1)$. The function f is said to be uniformly anti-Hölderian with exponent α, if there exists a constant C such that for all $(x,y) \in [0,1]^2$,

$$\sup_{(u,v) \in [x,y]^2} |f(u) - f(v)| \geq C|x-y|^\alpha \,.$$

The set of uniformly anti-Hölderian functions is denoted $I^\alpha([0,1])$.

Definition 3: Let $\alpha \in (0,1)$. If the function f both belongs to $\mathcal{C}^\alpha([0,1])$ and $I^\alpha([0,1])$ then f is said to be monoHölderian with exponent α.

B. Approximation properties

As already mentioned, only step 1 and 2 lead to approximations. If $f \in \mathcal{C}^\alpha([0,1])$, it is well–known [6], [11] that there exists a constant C (which depends on f but not on the scale j) such that

$$\|f - f_j\|_{L^\infty} \leq C 2^{-j\alpha},$$

whereas, if the function f is assumed to be uniformly monoHölderian, one deduces from [4] that there exists a constant C (which depends on f but not on the scale j) such that for any $\epsilon > 0$

$$\|f - f_j\|_{L^\infty} \geq C j^{-(2\alpha+\epsilon)} 2^{-j\alpha}.$$

Note that the last condition is much weaker than uniform anti–Hölderianity (see [4]) since it involves the modulus of continuity of $f - f_j$ on the whole interval $[0,1]$, whereas oscillations of uniformly anti-Hölderian functions can be bounded from below at any point. The approximation made at step 2 clearly does not depend on the regularity of function f, and we have

$$\|f_j - \tilde{f}_j\|_{L^\infty} \leq 2^{-M}.$$

C. Theoretical number of samples in the case of a monotonous function

If f is a monoHölderian function with exponent α, by definition there exists $C_1, C_2 > 0$ and for any scale $j \geq 0$ and $0 \leq k \leq 2^j - 1$

$$C_1 2^{-j\alpha} \leq \sup_{(u,v) \in [k/2^j, (k+1)/2^j]^2} |f(u) - f(v)| \leq C_2 2^{-j\alpha}.$$

If the function is additionally supposed to be monotonous, we have further that

$$\sup_{(u,v) \in [k/2^j, (k+1)/2^j]^2} |f(u) - f(v)| = |f((k+1)/2^j) - f(k/2^j)|.$$

Hence

$$\begin{aligned} C_1 2^{j(1-\alpha)} &\leq |f(1) - f(0)| \\ &= \sum_{k=0}^{2^j-1} |f(\frac{k+1}{2^j}) - f(\frac{k}{2^j})| \leq C_2 2^{j(1-\alpha)}. \end{aligned}$$

Such a signal crosses equi-spaced levels with quantum 2^{-M} at most $C2^{M+(1-\alpha)j}$ times. With our algorithm, we also take at most 2^j samples (since we use the initial samples). For large values of M (or small values of α), we indeed keep almost all the 2^j samples. Otherwise we can expect some reduction of the number of samples. For $C = 1$, the threshold is $M \simeq \alpha j$. Observe that the proof is based on the fact, that in the monotonous, we can estimate in a very simple way the oscillations

$$\sup_{(u,v) \in [k/2^j, (k+1)/2^j]^2} |f(u) - f(v)|$$

of the function. Of course in the general case, the situation can be much more complicated. Nevertheless, generic results in the sense of prevalence as stated in [2] are expected to hold. In what follows, we illustrate through numerical simulations what may happens.

D. Numerical simulations

1) Test functions: We test level crossing on two toy models: sample paths of fractional Brownian motion B_H and the Weierstrass function W_H. Here $H \in]0,1[$ is called the Hurst index. In each of these two cases, the smoothness properties of the function are well–known and related to the Hurst index.

The fractional Brownian motion (fBm) B_H is the unique Gaussian H–self-similar process with stationary increments. It can be defined from its covariance function

$$\mathbb{E}\big[B_H(x) B_H(y)\big] = \tfrac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right)$$

for all $(x,y) \in [0,1]^2$. For $H = 1/2$, we recover the classical Brownian motion. We recall that the sample paths of fBm are well-known to be almost surely continuous. Further, the Hurst index H of fBm is directly related to the roughness of its sample paths. More precisely, almost surely,

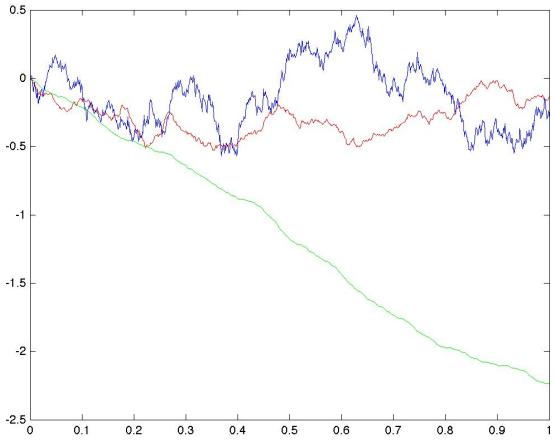

Fig. 1. Three realizations of fractional Brownian motions for $H = 0.5$ (blue plot), $H = 0.7$ (red plot), $H = 0.9$ (green plot)

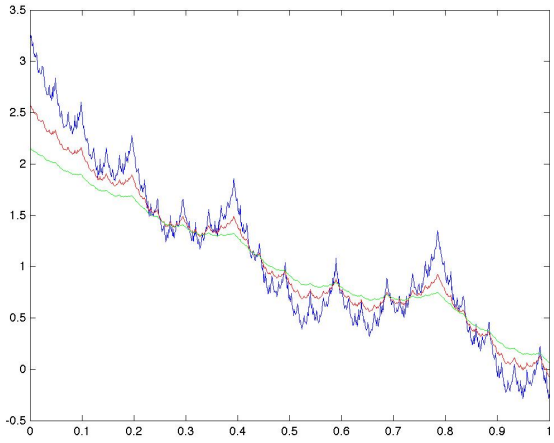

Fig. 2. The Weierstrass function for $H = 0.5$ (blue plot), $H = 0.7$ (red plot), $H = 0.9$ (green plot)

$B_H \in \mathcal{C}^{H-\varepsilon}([0,1]) \cap I^{H+\varepsilon}([0,1])$ (classical law of the iterated logarithm). Roughly speaking, a.s. for all $(x,y) \in [0,1]^2$, $\sup_{(u,v)\in[x,y]^2} |B_H(u) - B_H(v)| \sim |x-y|^H$. Figure 1 presents three realizations of sample paths of fractional Brownian motions for $H = 0.5$, $H = 0.7$, $H = 0.9$ and 1024 samples ($j = 10$).

We also use the Weierstrass function defined as

$$W_H(x) = \sum_{j=0}^{\infty} 2^{-jH} \cos(2^j x).$$

The Weierstrass function W_H is a classical example of monoHölderian function with exponent H as proved in [8]. Hence for all $(x,y) \in [0,1]^2$, $\sup_{(u,v)\in[x,y]^2} |W_H(u) - W_H(v)| \sim |x-y|^H$. Figure 2 present some graphs of Weierstrass functions for $H = 0.5$, $H = 0.7$, $H = 0.9$ and 1024 samples ($j = 10$).

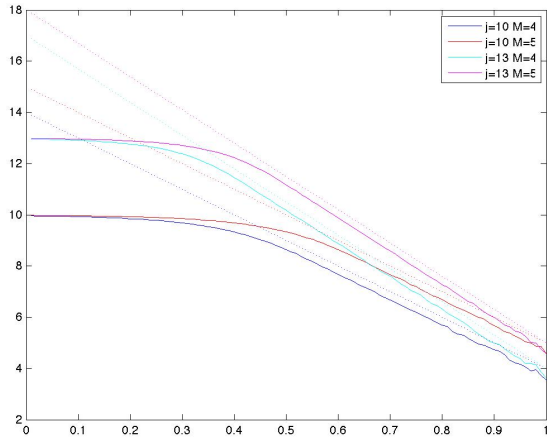

Fig. 3. Average number n of samples in terms of the Hurst number in the log scale ($\log_2(n)$ is represented on the y-axis). Four cases are plotted (solid lines) corresponding to $j = 10$ and 13 and $M = 4$ and 5. The dotted lines correspond to the worst case $M + (1-H)j$.

	$M = 4$	$M = 5$
$j = 10$	0.4	0.5
$j = 13$	~ 0.3	~ 0.4

TABLE I
"CRITICAL" VALUES OF THE HURST NUMBER.

2) Test cases: The tests are performed within the SPASS Matlab toolbox [1]. To generate fractional Brownian motions, we make use of the `genFBMJFC.m` function [3].

We use two values of j (10 and 13) and two values of M (4 and 5). These small values of M are sufficient for most mobile applications. Our output is the number of samples after decimation (Step 3). For the fractional Brownian motion, we perform 1000 realizations and average the number of samples obtained for each realization to obtain an average number n.

We perform this for values of the Hurst number H in the $]0,1[$ range and obtain the curves in Figure 3. We also plot the number of samples computed in the worst case (monotonous function i.e. maximum total variation) for $C = 1$: $2^{M+(1-H)j}$. The plots are given in the semi-log scale: $\log_2(n)$ and $M + (1-H)j$. This allows to distinguish the two regimes below some value of the Hurst number $H \sim M/j$ the algorithm more or less keeps all the original samples, above this value the decimation is efficient and yields a significant reduction of the number of samples.

For the different curves these "critical" values of H are given in Table I.

We perform the same tests on the Weierstrass function. The plots associated to fBm are much more regular because there are obtained by an averaging procedure. We expect that the critical value of M holds in an asymptotical way. Our results are then expected to improve when j tends to infinity.

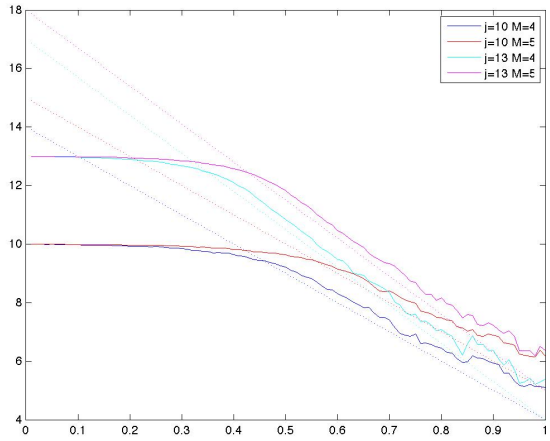

Fig. 4. Number of samples in terms of the Hurst number in the log scale ($\log_2(n)$ is represented on the y-axis). Four cases are plotted (solid lines) corresponding to $j = 10$ and 13 and $M = 4$ and 5. The dotted lines correspond to the worst case $M + (1 - H)j$.

V. Conclusion

We have first shown numerically that there is strong relationship between the smoothness properties of a signal and the number of samples that can be obtained by the crossing level algorithm presented in this paper. We also proved this result in the case of monotonous monoHölderian functions. We intent to address this problem in more general cases. It will be the purpose of a forthcoming paper.

Acknowledgment

This work has been supported by the MathSTIC Project OASIS of the Grenoble University. The authors want to thank Jean-François Coeurjolly for fruitful discussions.

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