Event-Triggered Stabilizing Controllers for Switched Linear Systems

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Abstract

We introduce an event-triggered algorithm for the stabilization of switched linear systems. We define a pseudo-Lyapunov function common to all the subsystems. The pseudo-Lyapunov function is compared, at every time instant, to an exponentially decreasing upper threshold. An event is generated when the two functions intersect, or when a new subsystem becomes active. The existence of a Lyapunov function common to all the subsystems is a key requirement of this method. Nevertheless, imposing this condition does not add to the complexity of the problem. Indeed, we formulate the problem in terms of Linear Matrix Inequalities, as a generalized eigenvalue problem. This formulation allows to simultaneously check for the existence of a common Lyapunov function and to obtain the optimal parameters to define the upper threshold.
We prove the stability of the system under the event-triggered control and we show that successive events are separated by a minimum interval of time.

Keywords: event-triggered control, switched linear system

1. Introduction

A switched system is a dynamical system composed of a finite set of subsystems with continuous dynamics, and a switching rule that determines

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which subsystem is active at a given time. The coexistence of continuous and discrete-time dynamics makes switched systems a subclass of hybrid systems. The difference between hybrid and switched systems though, is that the analysis of switched systems focuses on the continuous-time behavior, with discrete transitions between subsystems treated as isolated events [1].

Many physical systems can be modeled as switched systems. A non-exhaustive list of switched systems includes electronic circuits containing a switching device [2], [3], systems driven by several controllers, or systems with dynamics changing due to a damaged component [4].

Despite the modeling advantages that they offer, the stability analysis of switched systems is a challenging task. The reason is that the stability of each individual subsystem does not imply the stability of the entire system, under arbitrary switching. It has been proved in [5] and [6] that the existence of a quadratic Lyapunov function common to all subsystems (CQLF) guarantees the asymptotic stability of the switched linear system under an arbitrary switching rule. Even if this property is conservative, for a switched system can be stable when no common Lyapunov function exists, the existence of a CQLF can be directly verified by solving a set of Linear Matrix Inequalities (LMI) [7].

In this work, we propose an event-triggered stabilizing control algorithm for switched systems. In event-triggered control, the control signal is a piecewise constant function of time. Its value is updated only when the behavior of the system no longer satisfies some predefined performance criteria. This control method reduces the communications between the controller and the actuators, and saves actuating and computation resources.

The application of event-based control to the case of switched systems knows a growing interest among the control community. In [8], the authors synthesize an event-triggered dynamic controller for switched systems with time delays, based on the periodic event-triggered approach described in [9] and the dynamic event-triggering mechanism of [10]. In [11], the event-triggered control algorithm presented in [12] is applied to switched linear systems with model uncertainties, and multiple Lyapunov functions are used to prove stability. In [13], the authors propose an output-based event-triggered approach to the control of continuous-time switched systems. The event-triggering conditions rely on the squared error between the current and the most-recently sampled state.

In this work, we extend to switched systems the event-based control al-

gorithm that we developed in [14]. In this method, we define a pseudo-Lyapunov function (PLF), which, unlike a Lyapunov function that must strictly decrease in time, is allowed to increase locally, as long as it is globally decreasing and upper bounded. The control is updated when the PLF reaches a predefined threshold. For switched systems, we take the PLF as the common Lyapunov function of the subsystems.
For simplicity, we restrict our attention to switched systems with a common Lyapunov function but the proposed approach can be extended with a moderate effort to the case of Switched Quadratic Lyapunov functions [15], and polytopic systems [16].
This paper is organized as follows. In Section 2, we define all the terms and concepts involved in our discussion. We also give a formal definition of the event-triggered control problem. In Section 3, we give a detailed description of the event-triggered control algorithm, along with a proof that it asymptotically stabilizes a switched system under an arbitrary switching sequence. In the second part of Section 3, we prove that there is a minimum time between any two events. These proofs are more rigorous than the ones given in [14]. The final section provides a numerical example to show the performance of the algorithm, how it stabilizes a test system with a small number of control updates and with no Zeno behavior. We give an example with time switching and another with a state-dependent switching rule.

2. Problem Definition

Consider the switched linear system modeled as

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, and $u \in \mathbb{R}^m$ is the control signal. System (1) is also defined by the mapping $\sigma : [t_0, \infty) \to \mathcal{I}$, that we call the switching rule. The set $\mathcal{I} = \{1, 2, ..., I\}$ is a finite set, called the index set, such that every subsystem of the form

$$\dot{x}(t) = A_i x(t) + B_i u(t) \qquad \forall i \in \mathcal{I}, \tag{2}$$

is a realization of the switched system (1). A_i and B_i are constant matrices of appropriate dimensions and the pairs (A_i, B_i) are controllable for all $i \in \mathcal{I}$. Let x_0 be the initial state $x(t_0)$.

In what follows, we consider the switching rule to be arbitrary and either time-dependent or state-dependent, or both. Therefore, the notation $\sigma(t)$ does not imply that σ is only time dependent but refers to the realization of the rule σ at time t. In this framework, we refer to a change in the active subsystem as a jump of the switching rule.

We also suppose that the switched system (1) does not experience any chattering or Zeno behavior. This means that each subsystem remains active for at least some period of time $\tau_d \neq 0$. The time duration τ_d is called the dwell time. In the case of state-dependent switching, imposing a dwell time can be hard, as the switching occurs generally when the state crosses a set of surfaces. Instead of crossing, the state may slide along the surface, creating instantaneous switches. Fortunately, there exist several solutions to this problem. For example, in [17], a minimum dwell time is imposed by coupling state-switching and time-switching. Also, in Chapters 1 and 6 of [1], a dwell time is obtained through hysteresis switching, in which switching occurs when the state crosses a strip instead of a surface.

In event-triggered control, the control law $u(t)$ is a piecewise constant signal. Its value is updated when an event occurs in the system. In the case of switched systems, we identify two types of events

- the state of the system no longer satisfies some predefined performance criteria,
- there is a jump in the switching rule, and a different subsystem becomes active.

We denote by t_k, $k \in \mathbb{N}$, the time instants at which the events occur. The control law $u(t)$ is a state-feedback control and is scheduled as follows

$$u(t) = -K_i x(t_k), \quad t \in [t_k, t_{k+1}), \tag{3}$$

where the ith subsystem ($i \in \mathcal{I}$) is active at time t_k.

Let $\Delta_k x(t) = x(t_k) - x(t)$, then in the interval $[t_k, t_{k+1})$ and for all k, System (1) admits the solution

$$x(t) = e^{(A_i - B_i K_i)(t-t_k)} x(t_k) - \int_{t_k}^{t} e^{(A_i - B_i K_i)(t-s)} B_i K_i \Delta_k x(s) ds. \tag{4}$$

An important property of $x(t)$ is its Lipschitz continuity. Therefore, there exists a constant $L_x > 0$ such that

$$\|\Delta_k x(t)\| \leq L_x(t - t_k), \quad \forall k. \tag{5}$$

where $\|.\|$ designates both the Euclidean vector norm and the corresponding matrix norm. We can now give a formal statement of the problem.

Problem statement. Consider the switched linear system (1), composed of I subsystems that switch according to an arbitrary switching rule σ. Each subsystem is stabilizable through a state-feedback control law. The control law is updated when an event is triggered, and otherwise kept constant. Therefore, design an event-triggering condition, or a set of event triggering conditions that

- detect the system's failure to satisfy the performance criterion that we introduce in Section 3,
- detect a change in the active subsystem.

When an event occurs, the control law is updated such that

- the switched linear system (1) is asymptotically stable under an arbitrary switching rule,
- for any two consecutive events at t_k and t_{k+1}, there exists a minimum duration $\tau > 0$, such that $t_k - t_{k+1} \geq \tau$, for all $k \in \mathbb{N}$.

3. Event-triggered Control Algorithm

3.1. Algorithm Description

A sufficient condition for the stability of a switched linear system under arbitrary switching is the existence of a CQLF. The function $V : \mathbb{R}^n \to \mathbb{R}^+$ of the form

$$V(x(t)) = x(t)^T P x(t), \tag{6}$$

is a CQLF for the system (1) if and only if there exists a single positive-definite matrix P such that

$$(A_i - B_i K_i)^T P + P(A_i - B_i K_i) = -Q_i, \quad \forall i \in \mathcal{I}, \tag{7}$$

where Q_i are symmetric positive definite matrices. We also define $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ as the minimum and maximum eigenvalues of P, respectively. Also, $\lambda_{\min}(Q_i)$ and $\lambda_{\max}(Q_i)$ are the minimum and maximum eigenvalues of Q_i.

Even when each individual matrix $A_i - B_iK_i$ is Hurwitz for all $i \in \mathcal{I}$, a CQLF is not guaranteed to exist. In the literature, conditions for the existence of a CQLF were established for some particular classes of systems, such as the case when $A_i - B_iK_i$ are triangular matrices or when they commute. However, if the subsystems have a general structure, there is no algebraic condition on $A_i - B_iK_i$ to establish beforehand whether a CQLF exists [7], and the only way to determine the existence of a CQLF is to solve the system of equations (7).

In classical control, the Lyapunov function is strictly decreasing in time along system trajectories. In our event-triggered approach though, we relax this condition. In our case, $V(x)$ decreases for some time after the update of the control law. Then, when the control ceases to be effective, $V(x)$ increases until it reaches an upper threshold. At that moment the control law is updated, and $V(x)$ decreases again. In the case of switched systems, the control is also updated when a jump in the switching rule is detected. We refer to the function $V(x)$ as a pseudo-Lyapunov function.

Moreover, after an update of the control law, the time derivative of the PLF along the trajectories of the system, is given by

$$\frac{dV(x(t))}{dt}|_{t=t_k^+} = -x(t_k)^T Q_i x(t_k). \tag{8}$$

which is the same as the derivative of a classical Lyapunov function for all t. However, between two events, the derivative of the PLF differs from its classical counterpart and is given by

$$\frac{dV(x(t))}{dt} = -x(t)^T Q_i x(t) - 2\Delta_k x(t)^T K_i^T B_i^T P x(t), \quad t \in (t_k, t_{k+1}). \tag{9}$$

In [14], we have shown that in the case of linear time-invariant systems, if we take a positive, exponentially decreasing function as the upper threshold for the PLF [18], we can guarantee that the PLF decreases globally, despite being locally increasing. By globally decreasing, we mean that $V(x)$ is always

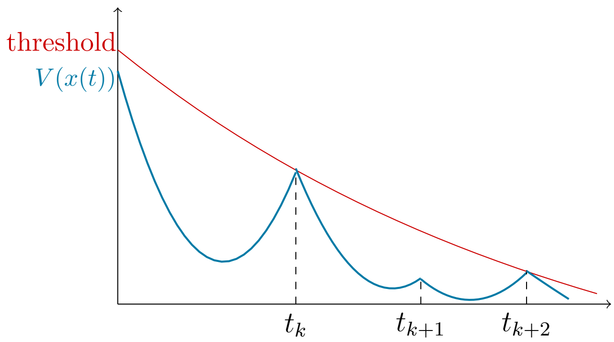


Figure 1: The Lyapunov-like function $V(x(t))$ in blue and the upper threshold in red.

below a strictly decreasing function. In the case of switched linear systems, where a CQLF exists, it is no different, as the CQLF is taken as a PLF, and the algorithm is designed in the same manner. The only added difficulty is to monitor the jumps of the switching rule.

Fig. 1 illustrates the behavior of the PLF. It shows that at times t_k and t_{k+2} an event is generated as the PLF hits the upper bound, and is pushed back below by control update, whereas the event at $t = t_{k+1}$ corresponds to a jump in the switching rule.

To explain how we obtain the exponential threshold, we recall that at $t = t_k$, when the control is updated, $V(x)$ decreases in time. Accordingly dV/dt is strictly negative at $t = t_k$, and we can define a negative upper bound on this quantity. More precisely, we look for a scalar $\lambda < 0$ such that the following constraint is satisfied

$$\frac{dV(x(t))}{dt}|_{t=t_k^+} \leq \lambda V(x(t_k)), \quad \forall k. \tag{10}$$

Yet, at $t = t_k^+$, after control update, the time derivative of $V(x(t))$ is given by Equation (8), therefore, from Equations (8) and (10)

$$-x(t_k)^T Q_i x(t_k) \leq \lambda x(t_k)^T P x(t_k), \quad \forall k, \ i = \sigma(t_k). \tag{11}$$

To ensure the fastest possible decay rate, we select λ as the maximum generalized eigenvalue of the pairs $(-Q_i, P)$, for all i, which is defined as [19]

$$\lambda_{\max}(-Q_i, P) \equiv \inf\{\lambda \in \mathbb{R} \mid -Q_i < \lambda P\}, \quad \forall i \in \mathcal{I}. \tag{12}$$

This value can therefore be found by solving the following optimization problem (where Q_i has been replaced by its expression from Equation (7))

$$\begin{aligned}
&\lambda_{\max} = \text{minimize } \lambda \\
&\text{subject to the LMI constraints} \\
&\quad (A_i - B_iK_i)^T P + P(A_i - B_iK_i) \leq \lambda P, \quad \forall i \in \mathcal{I} \\
&\quad P > 0,\ \lambda < 0.
\end{aligned} \tag{13}$$

Once we obtain the value of $\lambda_{\max} < 0$, we can extrapolate the solution of the differential inequality (10) to all $t \in [t_k, t_{k+1})$

$$V(x(t)) \leq V(x_k)e^{\lambda_{\max}(t-t_k)}. \tag{14}$$

Therefore, a suitable choice for the upper threshold function, denoted by $W(t)$, is

$$W(t) = W_0 e^{-\alpha(t-t_0)}, \tag{15}$$

where $W_0 > V(x_0)$ and $0 < \alpha \leq |\lambda_{\max}|$.

We can then define the execution times of the control law.

Definition 1. We define the time instants t_k, $k \in \mathbb{N}$, at which the control signal $u(t)$ is updated, as

$$t_k = \inf\{t > t_{k-1} \mid V(x(t)) \geq W(t) \textit{ or } \sigma(t) \neq \sigma(t_{k-1})\}, \tag{16}$$

where $V(x(t))$ and $W(t)$ are given by Equations (6) and (14), respectively.

3.2. Stability Results

Theorem 1. The control law, defined by Equation (3) and scheduled by the event-triggering condition given by Definition 1, renders System (1) asymptotically stable under arbitrary switching.

Proof. We show that the evolution of the PLF resulting from the control algorithm described above remains upper bounded by $W(t)$, for all t. And since $W(t)$ tends to zero as time tends toward infinity, so does $V(x(t))$, which in turns means that $x(t)$ converges to the zero equilibrium.

For $t = t_0$, $V(x_0) < W(t_0)$, by definition. Then, when $t > t_0$, we identify three cases:

1. Case $\sigma(t_k^-) = \sigma(t_k)$ (no jump) and $V(x(t_k)) = W(t_k)$
   When $t_{k-1} < t < t_k$, $V(x(t)) < W(t)$, by Definition 1.
   When $t = t_k$, $V(x(t_k)) = W(t_k)$, and after updating the control,

$$\frac{dV(x)}{dt}|_{t=t_k^+} = -x^T(t_k)Q_i x(t_k) \leq \lambda_{\max} V(x(t_k)). \tag{17}$$

   Since we select $\alpha < |\lambda_{\max}|$, then at time t_k, $\lambda_{\max}V(x(t_k)) < -\alpha W(t_k)$, and equation (17) becomes

$$\frac{dV(x)}{dt}|_{t=t_k^+} < -\alpha W(t_k) = \frac{dW(t)}{dt}|_{t=t_k} < 0. \tag{18}$$

   This proves that at $t = t_k^+$, $V(x(t))$ decreases faster than $W(t)$, and therefore remains below $W(t)$ for a certain time.

2. Case $\sigma(t_k^-) \neq \sigma(t_k)$ (jump) and $V(x(t_k)) < W(t_k)$
   Since $V(x)$ is a continuous function, an update of the control law at $t = t_k$ does not change the fact that $V(x(t)) < W(t)$. Since the update makes the time derivative dV/dt negative, $V(x)$ decreases at that instant, whether it was in a decreasing or increasing phase before the update.

3. Case $\sigma(t_k^-) \neq \sigma(t_k)$ and $V(x(t_k)) = W(t_k)$
   Since $V(x)$ is a common Lyapunov function for all the subsystems, this case is no different from Case 1 when $V(x(t_k)) = W(t_k)$ with no jump of the switching rule.

□

3.3. Minimum Inter-Event Time

The event-triggered control algorithm also needs to guarantee a minimum time lapse between any two successive events. If no such time exists, we could end up with an infinite number of updates in a finite interval of time, a situation known as the Zeno phenomenon.

Theorem 2. Let $T > t_0$ arbitrarily large, t_k and t_{k+1} two consecutive time instants in $[t_0, T]$ given by Definition 1. Then there exists a minimum time $\tau > 0$ such that $t_{k+1} - t_k \geq \tau$, on the finite interval $[t_0, T]$.

Remark 1. The parameter T, that can be chosen arbitrarily large, allows us to prove the existence of an inter-event time as it offers many advantages

- We avoid the risk of obtaining events due to the switching rule that are arbitrarily close to the events due to the PLF for large times. As a result, there always exists a minimum time τ, either between two intersections or between a jump and an intersection.

- It allows us to fix a lower bound on the exponential threshold.

Proof. The proof of Theorem 2 depends on the nature of each event in a pair of consecutive events. We identify the following cases

- t_k and t_{k+1} are due to an intersection between the PLF and the threshold: this possibility is covered in Case 1, below.

- t_k is due to a jump in the switching rule, while t_{k+1} is due to an intersection: this is covered in Case 2.

- t_k and t_{k+1} are both due to a jump: in this case the minimum inter-event time is τ_d, the dwell time.

- t_k is the result of an intersection and t_{k+1} is due to a jump: since there is a finite number of jumps, this can occur only a finite number of times. Therefore, the minimum of these finite delays is nonzero, and there exists a minimum inter-event time. However, we cannot give an estimation of this inter-event time.

1. Case $V(x(t_k)) = W(t_k)$
   To prove the existence of τ, we need to show that $V(x(t))$ decreases faster than $W(t)$ for some time after an update of the control law, such that no other intersection is possible. To prove this fact, we first find lower and upper bounds on $\|x(t_k)\|$. Afterward, we use these bounds to show that $dV(x(t))/dt < dW/dt$, in some interval $[t_k, t_k + \tau)$.

   - Lower and upper bounds on $\|x(t_k)\|$:
     At $t = t_k$, $V(x(t_k))$ admits the following bounds

$$\lambda_{\min}(P)\|x(t_k)\|^2 \leq V(x(t_k)) \leq \lambda_{\max}(P)\|x(t_k)\|^2. \tag{19}$$

Since $V(x(t_k)) = W(t_k)$, and $W(t)$ is an exponentially decreasing function, such that $W(T) < W(t_k) < W_0$, we can write

$$\sqrt{\frac{W(T)}{\lambda_{\max}(P)}} \leq \|x(t_k)\| \leq \sqrt{\frac{W_0}{\lambda_{\min}(P)}}. \tag{20}$$

We denote $M = \sqrt{W_0/\lambda_{\min}(P)}$.

- Proving that $dV/dt < dW/dt$ in an interval $[t_k, t_k + \tau)$:
  When $t \in (t_k, t_{k+1})$, the time derivative of $V(x(t))$ is given by Equation (9), which can be re-written in terms of $x(t_k)$ and $\Delta_k x(t)$ as

  $$\begin{aligned}\frac{dV(x(t))}{dt} = &- x(t_k)^T Q_i x(t_k) + x(t_k)^T Q_i \Delta_k x(t) \\ &+ \Delta_k x(t)^T (Q_i - 2K_i^T B_i^T P) x(t_k) \\ &+ \Delta_k x(t)^T (2K_i^T B_i^T P - Q_i) \Delta_k x(t).\end{aligned}$$

  We use Equations (17), (20), and the Lipschitz continuity of $x(t)$ with the Lipschitz constant given by Equation (5), to find an upper bound on $dV(x(t))/dt$

  $$\begin{aligned}\frac{dV(x(t))}{dt} \leq \; &\lambda_{\max} W_0 e^{-\alpha(t_k - t_0)} + M \lambda_{\max}(Q_i) L_x (t - t_k) \\ &+ L_x M \|Q_i - 2K_i^T B_i^T P\| (t - t_k) \\ &+ \|2K_i^T B_i^T P - Q_i\| L_x^2 (t - t_k)^2.\end{aligned} \tag{21}$$

  Equation (21) is of the form

  $$\frac{dV(x(t))}{dt} \leq \lambda_{\max} W_0 e^{-\alpha(t_k - t_0)} + C_1 (t - t_k) + C_2 (t - t_k)^2.$$

  Re-writing the derivative of $W(t)$ as

  $$\frac{dW(t)}{dt} = -\alpha W_0 e^{-\alpha(t_k - t_0)} e^{-\alpha(t - t_k)},$$

  It is sufficient to show that for $t \in (t_k, t_k + \tau)$,

  $$\begin{aligned}\lambda_{\max} W_0 e^{-\alpha(t_k - t_0)} + C_1 (t - t_k) + &C_2 (t - t_k)^2 \\ &< -\alpha W_0 e^{-\alpha(t_k - t_0)} e^{-\alpha(t - t_k)}.\end{aligned} \tag{22}$$

Dividing both sides of Equation (22) by the quantity $-\alpha W_0 e^{-\alpha(t_k - t_0)}$ yields

$$\frac{|\lambda_{\max}|}{\alpha} > \frac{C_1(t-t_k)}{\alpha W_0 e^{-\alpha(t_k-t_0)}} + \frac{C_2(t-t_k)^2}{\alpha W_0 e^{-\alpha(t_k-t_0)}} + e^{-\alpha(t-t_k)} =: f_k(t),$$

This equation is satisfied at $t = t_k$, as $f_k(t_k) = 1$ and $|\lambda_{\max}|/\alpha > 1$. The function $f_k(t)$ is Lipschitz continuous uniformly in k. Therefore, in a sufficiently small interval $[t_k, t_k + \tau)$, we can guarantee that

$$\frac{|\lambda_{\max}|}{\alpha} > f_k(t).$$

and τ is a uniform minimum inter-event time for all k.

2. Case $\sigma(t_k^-) \neq \sigma(t_k)$ and $V(x(t_k)) < W(t_k)$
For this type of events, there is nothing we can say about the rate of decay of $V(x(t))$ with respect to the decay rate of $W(t)$. However, depending on how far $V(x(t_k))$ is from $W(t_k)$, we can show that some time has to pass before their next intersection. We can analyze this time lapse based on the difference between $V(x(t_k))$ and the quantity $W(T)/2$, which falls into two categories.

   - Case $V(x(t_k)) \geq W(T)/2$.
     Since $V(x(t_k)) < W(t_k)$, we show that $V(x(t))$ changes slowly enough, so that no intersection between the PLF and the threshold is possible until some time passes. For this we show that the derivative of $V(x(t))$ remains bounded from above for some amount of time. This, in turn, provides an estimation of the time τ.

     In the case when $V(x(t_k)) \geq W(T)/2$, Equation (19) can be rewritten as

     $$\sqrt{\frac{W(T)}{2\lambda_{\max}(P)}} \leq \|x(t_k)\| \leq M.$$

     We denote $\mu = \sqrt{W(T)/2\lambda_{\max}(P)}$.
     When $t_k < t < t_{k+1}$, $x(t)$ is given by the integral equation (4). Also, in the interval $[t_0, T]$ and due to the Hurwitz property of $A_i - B_i K_i$, there exist two positive constants γ and Γ, such that

     $$\gamma \|x(t_k)\| \leq \|e^{(A_i - B_i K_i)(t-t_k)} x(t_k)\| \leq \Gamma \|x(t_k)\|.$$

Besides, since $x(t)$ is Lipschitz continuous, we can deduce the following lower bound on $\|x(t)\|$,

$$\|x(t)\| \geq \gamma\ \mu - \Gamma\|B_i K_i\|\frac{L_x}{2}(t-t_k)^2.$$

So, if we select a sufficiently small τ_1, when $t \in [t_k, t_k+\tau_1)$, there exists $\varepsilon > 0$, such that

$$\|x(t)\| \geq \varepsilon.$$

Using this lower bound yields an upper bound on the first term of Equation (9)

$$-x(t)^T Q x(t) \leq \lambda_{\min}(Q_i)\varepsilon =: -2\beta.$$

The second term admits an upper bound

$$\| - 2\Delta_k x(t)^T K_i B_i P x(t)\| \leq 2L_x(t-t_k)\|K_i B_i P\|M,$$

which can be rendered smaller than β by choosing t from an interval $[t_k, t_k+\tau_2)$ with $\tau_2 \leq \tau_1$ and τ_2 small enough. Therefore, for $t \in [t_k, t_k+\tau_2)$, $dV(x(t))/dt \leq -\beta$, and $V(x(t))$ is strictly decreasing during this interval.

- Case $V(x(t_k)) < W(T)/2$.
  In this case, we cannot find a lower bound on $\|x(t)\|$, and thus we cannot estimate the time during which $dV(x(t))/dt$ decreases. However, we know that

  $$V(x(t_k)) < W(T)/2 < W(T) < W(t_k).$$

  Due to the large gap between $V(x(t_k))$ and $W(t_k)$, and to the Lipschitz continuity of $V(x(t))$, an intersection between $V(x(t))$ and $W(t)$ is not possible until some time τ has elapsed, as shown on Figure 2.

  Additionally, before an intersection with $W(t)$ is possible, $V(x(t))$ has to go first through the value $W(T)/2$ and then $W(T)$ (see Figure 2). If we assume that $V(x(t))$ increases linearly with the maximum possible rate, we can estimate the time it takes for $V(x(t))$ to go from $W(T)/2$ to $W(T)$, as a lower bound on τ. Let τ_3 be this lower bound.

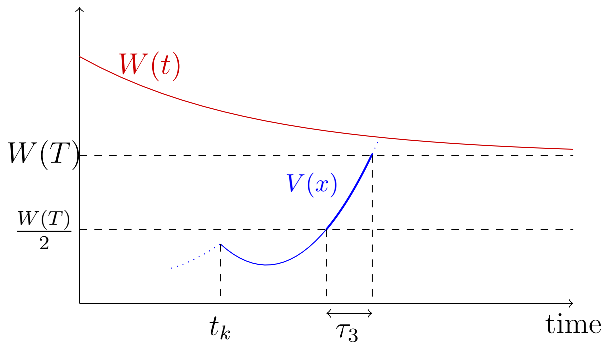


Figure 2: Illustration of the existence of an inter-event time when $V(x(t_k)) < W(T)/2$.

To determine the maximum possible rate, we need to find an upper bound on $|dV(x(t))/dt|$. We first re-write Equation (9) in the following form

$$\frac{dV(x(t))}{dt} = x(t)^T(2K_i^T B_i^T P - Q_i)x(t) - 2x(t_k)^T K_i^T B_i^T P x(t).$$

When $W(T)/2 \leq V(x(t)) \leq W(T)$

$$\|x(t)\| \leq \sqrt{\frac{W(T)}{\lambda_{\min}(P)}} =: \eta.$$

Then, we recall that as $V(x(t_k)) < W(T)/2$

$$\|x(t_k)\| < \sqrt{\frac{W(T)}{2\lambda_{\min}(P)}} = \frac{\sqrt{2}}{2}\eta.$$

The maximum possible rate is then

$$|\frac{dV(x(t))}{dt}| \leq (\|2K_i^T B_i^T P - Q_i\| + \sqrt{2}\|K_i^T B_i^T P\|)\eta^2 =: \psi.$$

Therefore, the time it takes $V(x(t))$ to reach $W(T)$ from $W(T)/2$ is given by the equation

$$\psi\tau_3 \approx \frac{W(T)}{2}.$$

A lower bound on τ is then

$$\tau \geq \frac{W(T)}{2\psi}.$$

$\square$

4. Numerical Example

4.1. Time-Dependent Switching

Consider the following second order switched system with three subsystems [20]

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.13 & -0.25 \\ 0.39 & -1.17 \end{bmatrix}, & B_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.35 & -0.42 \\ -0.43 & 0.01 \end{bmatrix}, & B_2 &= \begin{bmatrix} -3.925 \\ -2.11 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -1.58 & 0.01 \\ -0.91 & 0.71 \end{bmatrix}, & B_3 &= \begin{bmatrix} 0.02 \\ -0.08 \end{bmatrix}, \end{aligned}$$

with $x_0 = [1 \quad -0.2]^T$. By using the following feedback gains

$$\begin{aligned} K_1 &= \begin{bmatrix} 0.499 & -0.0074 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -1.0146 & 1.0493 \end{bmatrix}, \\ K_3 &= \begin{bmatrix} 1.7845 & -16.1789 \end{bmatrix}, \end{aligned}$$

we can make the three subsystems individually stable. They also allow us to find a CQLF with

$$P = \begin{bmatrix} 3.7698 & -3.7031 \\ -3.7031 & 4.4162 \end{bmatrix},$$

and decay rate $\lambda_{\max} = -0.5701$. We have obtained P and $\lambda_{\max}$ using the 'gevp' function of MATLAB. We choose $\alpha = 0.52$ s^{-1} and $W(t_0) = 5.928$. We have simulated the system for 30 seconds with a sampling time of 0.001 s. We have chosen a small sampling time to mimic the continuous-time behavior of the system. In addition, the active subsystem is chosen at random every $T_\sigma = 1.5$ s. The switching sequence is shown in Figure 3.

Figure 4 shows the time evolution of the two states $x_1(t)$ and $x_2(t)$. The two states converge to equilibrium around $t = 10$ s.

Figure 5 shows the event-triggered control law $u(t)$. The control has been updated 27 times, 14 times due to an intersection between $V(x(t))$ and $W(t)$,

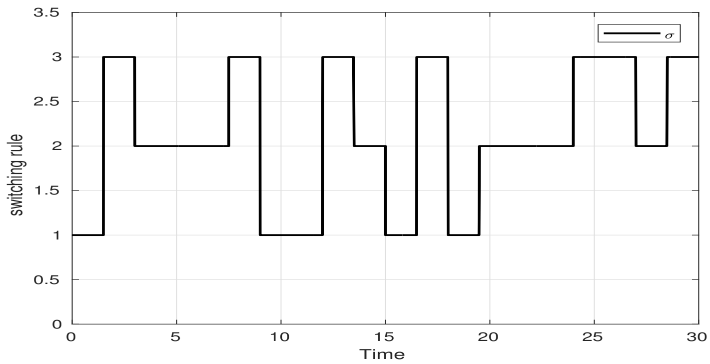


Figure 3: The switching sequence.

and 13 times due to a jump in the switching rule. Considering that the total number of simulation steps is $30,000$, we have decreased the communications between the controller and the plant by a factor of $1/1000$.

Figure 6 represents the PLF $V(x(t))$ and the upper threshold $W(t)$. We display the first 11 seconds only, for beyond that time, $W(t)$ and $V(x(t))$ approach zero, and it becomes harder to spot the events. We can notice the increases and decreases of the PLF, and the global convergence to zero. We can see the updates due to a jump in the switching rule, for example at $t = 1.5$ s. Examples of updates due to an intersection between the PLF and the threshold occur at $t = 2.83$ s and $t = 3.64$ s.

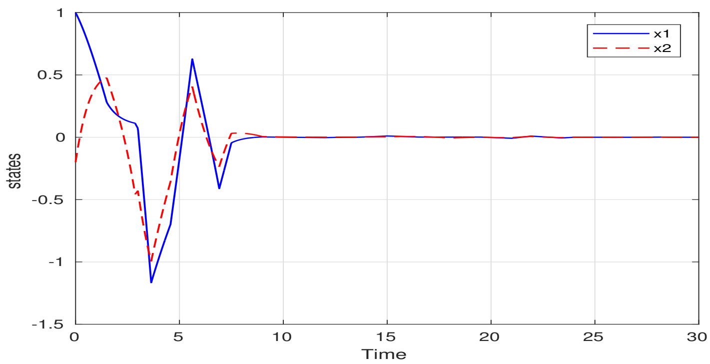


Figure 4: The time evolution of the states of the switched system.

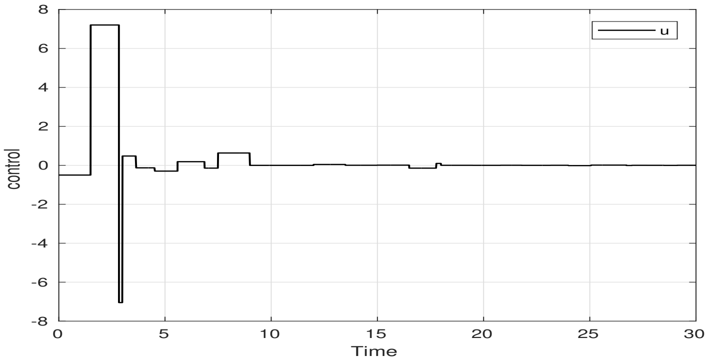


Figure 5: The event-based control signal.

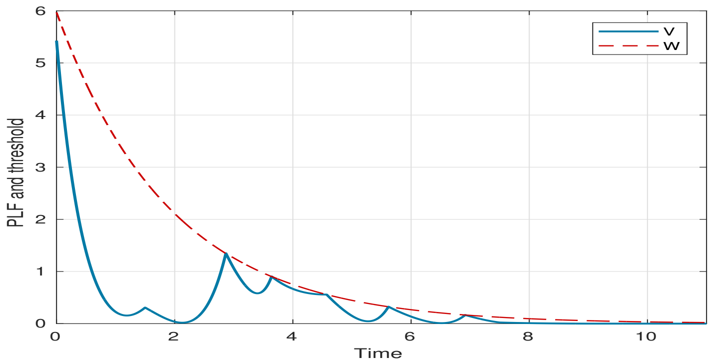


Figure 6: The Lyapunov function (in blue) and the exponential threshold (in red).

To see the events for the entire simulation window, we display the distribution of events in Figure 7. It shows events that are unevenly distributed in time. Successive events are generally far apart, but can also be clustered together. This distribution further emphasizes the philosophy of event-triggered control to give attention to a system when most needed.

Remark 2. When working in discrete-time, the instant at which $V(x(t)) = W(t)$ cannot be detected. For this reason, at $t = t_k$, we re-adjust $W(t)$ as (see [14] for more details)

$$W(t_k) = V(x(t_k)),$$

and for all $t \in (t_k, t_{k+1})$, $k > 0$,

$$W(t) = V(x(t_k))e^{-\alpha(t-t_k)}.$$

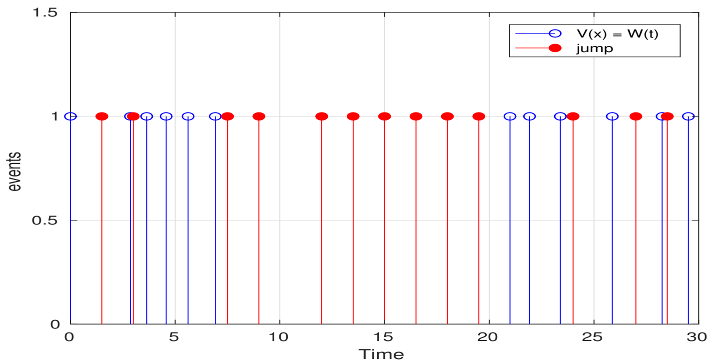


Figure 7: The events due to intersections (in blue) and to jumps (in red).

4.2. State-Dependent Switching

In this example, we examine the event-triggered control strategy when the switching rule is state-dependent. For this, we consider the following second order switched linear system, with two subsystems [21].

$$A_1 = \begin{bmatrix} -0.5 & 0 \\ 0.1 & 0.4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.2 & 1 \\ 0 & 0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix},$$

with $x_0 = [0.15 \quad -0.25]^T$.
For the feedback gains

$$K_1 = \begin{bmatrix} 0.0737 & -0.6632 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 1.7797 & 1.5441 \end{bmatrix},$$

there exists a CQLF with

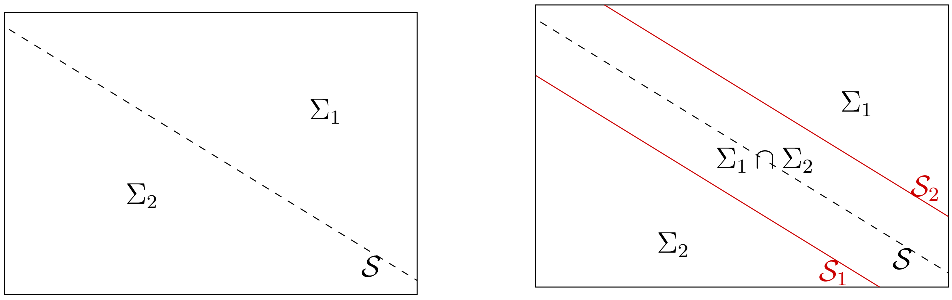
$$P = \begin{bmatrix} 0.6659 & 0.1481 \\ 0.1481 & 0.4937 \end{bmatrix}, \quad \lambda_{\max} = -0.514.$$

Hence, we select $\alpha = 0.513$ s^{-1} and $W(0) = 1.1V(x_0) = 0.0382$. We also use the same simulation time and sampling period as in the previous example.

To construct a state-dependent switching rule, we divide the state space into two regions, Σ_1 and Σ_2, separated by the surface $\mathcal{S}$, as shown in Figure 8a. Thus, when the state is in region Σ_1, subsystem 1 is active, whereas in region Σ_2, subsystem 2 is active.

When the surface $\mathcal{S}$ is a sliding surface, the state slides along $\mathcal{S}$ towards the origin. However, in a discrete-time simulation, the state keeps crossing to one side of $\mathcal{S}$ or the other during the sliding mode, creating switches at every instant. Such a situation contradicts our assumption that each subsystem must remain active for at least some duration τ_d. To solve this problem, we use the strategy in [1] called hysteresis switching.

In hysteresis switching, the surface $\mathcal{S}$ is off-set to the right and to the left, to define two new surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. This results in a strip (see Figure 8b) between the two new surfaces, intersecting both regions Σ_1 and Σ_2. This way, no switching occurs when either surface is crossed until the state leaves the common region $\Sigma_1 \cap \Sigma_2$.



(a) Regions of the state-space (b) Off-setting the switching surface

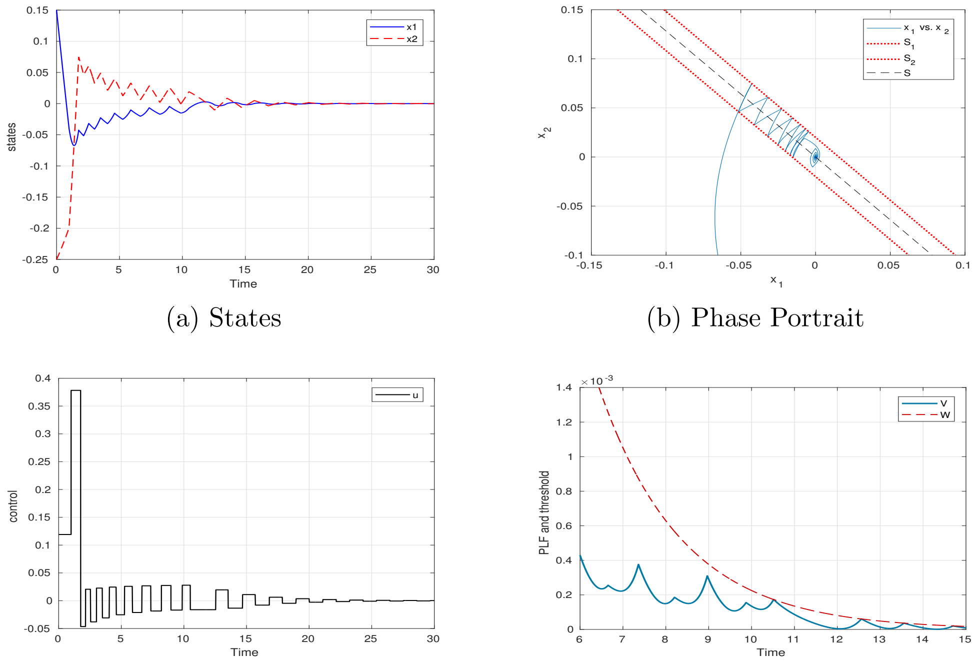
Figure 8: State-dependent and hysteresis switching.

In our example, we choose

$$
\begin{aligned}
(\mathcal{S}) :&\ \ x_2 = -1.2825x_1, \\
(\mathcal{S}_1) :&\ x_2 = -1.2825x_1 - 0.02, \\
(\mathcal{S}_2) :&\ x_2 = -1.2825x_1 + 0.02.
\end{aligned}
$$

Figure 9a shows the evolution of the states of the system with time. We see that the states eventually tend to equilibrium, proving the effectiveness of our approach. However, the states undergo an oscillation phase in transient-time. This is due to the fact that in the transient regime, the system experiences many switches as shown in the phase portrait of Figure 9b.

From Figure 9b, we also verify the effects of hysteresis switching as the state bounces between $\mathcal{S}_1$ and $\mathcal{S}_2$, thus allowing for a dwell time. When the state reaches a vicinity of the equilibrium, it remains inside a ball centered at the origin and switching stops.



(a) States

(b) Phase Portrait

(c) Control

(d) Zoom on the PLF

Figure 9: Simulation results for state-dependent switching.

Figure 9c shows the event-triggered control law, which has been updated 34 times, 20 times due to an intersection between the PLF and threshold and 14 times due to a jump in the switching rule. Figure 9d shows the PLF between $t = 6$ s and $t = 15$ s. Figure 9d reflects what is seen on the phase portrait of Figure 9b, as the events in transient-time are mostly due to a jump in the switching rule, whereas events in stead-state are due to an intersection between the PLF and the threshold.

5. Conclusion

We have presented an event-triggered control method to stabilize a linear switched system. We have shown that unlike classical control, where the control law is sampled at a high frequency, an event-based implementation achieves asymptotic stability while reducing drastically the number of samples.
We have also presented a way to obtain the common Lyapunov function of the system and its upper bound by solving a unique optimization problem. The generalized eigenvalue problem can be solved by software solutions and widely spread numerical methods.
As a perspective, we suggest an extension to the case of switched systems with Multiple Lyapunov Functions or Piecewise Quadratic Lyapunov Functions.

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