On the Cauchy problem for some systems occurring in nonlinear optics.

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Abstract

We study the Cauchy problem for two systems of equations (Maxwell-Debye and Maxwell-Bloch) describing laser-matter interaction phenomena. We show that these problems are locally in time well-posed for initial data in different Sobolev spaces. In the case of Maxwell-Debye system, which contains some delay term, we study the limit of the solutions when this delay tends to 0. We also consider an adiabatic approximation of Maxwell-Bloch system.

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Introduction

Our goal is an analytical study of equations governing the propagation of light through a medium which interacts with the electromagnetic field corresponding to this light wave. Thus we study two systems of equations which have a similar mathematical structure : Maxwell-Debye system

$$\begin{cases} \left(\dfrac{\partial}{\partial z} + \dfrac{n_0}{c}\dfrac{\partial}{\partial t}\right) A - \dfrac{i}{2k}\nabla_1^2 A + i\dfrac{\omega_0}{c}\delta n A = 0, \\ \tau\dfrac{\partial \delta n}{\partial t} + \delta n = n_2 |A|^2. \end{cases}$$

which describes the interaction of an electromagnetic wave with a nonresonant medium which has a relevant response time and Maxwell-Bloch system

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c}A = \dfrac{i\omega}{2\epsilon_0 c}L, \\ \\ \dfrac{\partial L}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar}AN, \\ \\ \dfrac{\partial N}{\partial t} + \gamma_{11}(N - N_0) = \dfrac{2i}{\hbar}(A^* L - AL^*). \end{cases}$$

which describes the interaction of an electromagnetic wave with a resonant medium which is constituted by a gas of two-level atoms.

There exists already various articles about a simpler version of Maxwell-Bloch equations, which consists in neglecting the Laplacian (∇_1^2) with respect to the x and y space variables. Thus we neglect the transversal variations of the field, i.e. we consider a (1+1)-dimensional problem.
In an article of Constantin, Foias and Gibbon [4], this (1+1)-dimensional equation is studied for periodic boundary conditions with respect to z. Then the system happens to be a nonlinear hyperbolic one. They study the global existence of solutions in L^2 and construct a finite dimensional universal attractor in this space. This attractor is constituted by $\mathcal{C}^\infty$ functions. This system has the complex Lorenz system as sub-system, when we restrict ourselves to solutions with no dependence with respect to the space variable z. This enables us to have an inkling of the complexity of the dynamics of these

equations.
Lega, Moloney and Newell [6] derive from the same Maxwell-Bloch equations a complex equation of Swift-Hohenberg type which capture the main features of the laser dynamics. They also analyze the stability of travelling wave solutions.
Concerning numerical computations, we cite Martín, Pérez-García, Guerra, Tirado and Vázquez [7] who developped a linearly implicit finite difference scheme for the Maxwell-Bloch equations (with no z-dependence) using a multigrid technique.

We specify in a first part the derivation of both models. The following parts are devoted to the study of the Cauchy problem. We are dealing with solutions which are bounded with respect to variables t (on a certain interval) and z and belong to Sobolev spaces with respect to transverse space variables, x and y. This is consistent with the physical point of view : the propagation of a laser beam which is regular and localized with respect to the transverse space variables. In the second part we show that the Cauchy problem for the Maxwell-Debye equations is locally well posed in H^s for $s > 1$ (smooth solutions) and next in H^1 and L^2 (weak solutions). In the case when $s \geq 1$, we also show that, as the delay τ tends to 0, the solutions to Maxwell-Debye equations tend in H^s to that of the Schrödinger equation which is the formal limit. The third part regroups a few results about Maxwell-Bloch equations. We begin with the study of an adiabatic approximation to finish with the Cauchy problem in H^s, $s > 1$ for the whole system.

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1 The equations of nonlinear optics.

We are dealing with models for the description of the propagation of light in an active resonant medium. Since the medium has a huge number of degrees of freedom, we restrict ourselves to a low number of them thanks to various assumptions. The electric field is supposed to be a collection of almost monochromatic wavetrains and we assume that the degrees of freedom that directly or undirectly resonate with the electric field are the only ones to have a longtime influence. The ingredients of the modelling may be found in Newell and Moloney's book [8].

We first begin with the derivation of Bloch equations that describe the dynamics of oscillators (excited atoms) of the matter.
The state of matter is described by the wave function ψ and the hamiltonian operator H whose eigenvalues are the (quantified) energy levels and eigenfunctions are the basis states.
In the unperturbed state, the hamiltonian is denoted by H_0, the energy levels by $E_j = \hbar\omega_j$ with eigenfunctions ψ_j respectively, verifying $H_0\psi_j = \hbar\omega_j\psi_j$.
$\{\psi_j\}$ is an orthonormal basis of the phase space and we may choose eigenfunctions so that moreover $\displaystyle\int \vec{R}\psi_j(\vec{R})d\vec{R} = 0$ and $\displaystyle\int \vec{R}\psi_j\psi_j^* d\vec{R} = 0$.
We want to compute the polarization vector induced by the field $\vec{E}$. As a general rule, it is given by

$$\vec{P} = n_a e \int \vec{R}\psi\psi^* d\vec{R}$$

where n_a is the volume density of atoms, and e the electric charge.

The first step consists in using Schrödinger equation. We suppose that ψ is solution to

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

where H is the sum of H_0 and of the pertubation potential $\delta V = -e\displaystyle\int \vec{E}.d\vec{R}$.
As $\vec{E}$ changes very little over atomic distances, we will consider that $\delta V = -e\vec{E}.\vec{R}$.

The second step consists in finding how unperturbed states read. To that aim we suppose that $\psi(\vec{R},t) = \sum_{j=1}^{N} a_j(t)\psi_j(\vec{R})$. We obtain in a straightforward way that if $H = H_0$ then $a_j(t) = a_j(0)e^{-i\omega_j t}$.

The third step is the computation of the polarization vector. We have

$$\vec{P}_{atom} = e\int \vec{R}\psi\psi^* d\vec{R} = \sum_{jk} \rho_{jk}\vec{P}_{kj} = \mathrm{Tr}\,\vec{p}\rho$$

where the density matrix element $\rho_{jk} = a_j a_k^*$ depends on time and the dipolar matrix element $\vec{p}_{jk} = e\int \vec{R}\psi_j^*\psi_k d\vec{R}$ not. We clearly have $\vec{P} = n_a\vec{P}_{atom}$. There remains to write dynamical equations for ρ_{jk}.

The fourth step is the derivation of "raw" Bloch equations (in that they is too many of them). Using the particular form of H, we obtain

$$\frac{\partial a_k}{\partial t} = -i\omega_k a_k + \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N} \vec{p}_{kl} a_l$$

and therefore for the density matrix elements

$$\frac{\partial \rho_{jk}}{\partial t} = -i(\omega_j - \omega_k)\rho_{jk} + \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N} \vec{p}_{jl}\rho_{lk} - \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N} \vec{p}_{lk}\rho_{jl}.$$

The fifth step is devoted to the simplification of these equations. We know how to solve the former equations if we neglect terms containing $\vec{E}$, but this may be only accomplished if they are actually neglectible. In practice, we may neglect all $\frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N} \vec{p}_{jl}\rho_{lk} - \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{N} \vec{p}_{lk}\rho_{jl}$ terms but those having frequences close to $\omega_{jk} = \omega_j - \omega_k$. Thus a large number of terms may ne neglected, their sum is nonetheless non neglectible. The gradual loss from the few modes we are interested in to the larger number of other modes is modellized by adding a term reading $-\gamma_{jk}\rho_{jk}$ ($\gamma_{jk} > 0$) in the equations for the n energy levels we keep. We will not study the other equations. Thus we performed a simplification of the equations as well as a reduction of their number.

The sixth step consists in identifying the possible resonances.
There may be some direct resonance, i.e. frequency ω in $\vec{E}$ is near to ω_{jk}

($\vec{E}.\sum \vec{p}_{jk}\rho_{kk}$ type of products).
There may be D.C. rectification ($\vec{E}.\sum \vec{p}_{jl}\rho_{lj}$ terms in which frequency $-\omega$ in $\vec{E}$ cancels the frequency $\omega_{lj} \sim \omega$ in ρ_{lj}, inducing a longtime cumulative effect on ρ_{jj}).
Last there may be some parametric resonance as one of the binary combinations of field difference frequencies $\pm\omega_r$ with dipolar difference frequencies $\pm\omega_{lk}$ are equal to other difference frequencies $\pm\omega_{jk}$.

Thus for the n levels we consider, we have

$$\frac{\partial \rho_{jk}}{\partial t} = -i(\omega_j - \omega_k)\rho_{jk} + \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{n} \vec{p}_{jl}\rho_{lk} - \frac{i\vec{E}}{\hbar}.\sum_{l=1}^{n} \vec{p}_{lk}\rho_{jl} \tag{1.1}$$

and the polarization vector is given by

$$\vec{P} = n_a \mathrm{Tr}\, \vec{p}\rho. \tag{1.2}$$

Last we derive Maxwell's equation governing the envelope of $\vec{E}$.
Maxwell's equations are

$$\begin{aligned} \vec{\nabla}.\vec{D} &= \rho, && \text{electric Gauss equation,} \\ \vec{\nabla}.\vec{B} &= 0, && \text{magnetic Gauss equation,} \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, && \text{Faraday equation,} \\ \vec{\nabla} \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{\jmath}, && \text{Ampère equation.} \end{aligned}$$

$\vec{E}$ is the electric intensity field, $\vec{B}$ the magnetic induction field, $\vec{D}$ the electric induction field and $\vec{H}$ magnetic intensity field. These fields are connected through the relations $\vec{B} = \mu\vec{H}$ and $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ where ϵ and μ are the dielectric constant and the permittivity of the matter respectively. We consider here that the electric charge density ρ and the electric current density $\vec{\jmath}$ are zero and that μ is constant and equal to $\mu_0 = \dfrac{1}{\epsilon_0 c^2}$ where c is the speed of light in the vacuum and ϵ_0 the dielectric constant of the vacuum.
Then we easily obtain

$$\vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial^2 t} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 \vec{P}}{\partial^2 t}. \tag{1.3}$$

2 Maxwell-Debye equations.

2.1 Modelization.

The presence of an electromagnetic field induces a variation of the medium refraction index, which we assume here to be nonresonant and with a non neglectible response time τ. Let $\vec{D} = \epsilon_0 n(\omega)\vec{E}$ be the deplacement electric field, where $n(\omega)$ is called index of the medium and reads

$$n(\omega) = n_0(\omega) + \delta n(E).$$

The equation for δn is Debye's equation, namely

$$\tau \frac{\partial \delta n}{\partial t} + \delta n = n_2 |E|^2.$$

This evolution law is relatively intuitive if we notice that for a medium that reacts "instantaneously", we have $n(\omega) = n_0(\omega) + n_2|E|^2$. We split the polarization vector in a linear polarization $\vec{P}_L = \epsilon_0(n_0^2(\omega) - 1)\vec{E}$ and a nonlinear one $\vec{P}_{NL} = 2\epsilon_0 n_0 \delta n(E)\vec{E}$. In the case of an unidirectional wave, we may set $\vec{E} = \hat{e}A(\vec{r}, t)e^{i(kz-\omega t)} + c.c.$ and (1.3) becomes

$$\begin{cases} \left(\dfrac{\partial}{\partial z} + \dfrac{n_0}{c}\dfrac{\partial}{\partial t}\right) A - \dfrac{i}{2k}\nabla_1^2 A + i\dfrac{\omega_0}{c}\delta n A = 0, \\ \tau \dfrac{\partial \delta n}{\partial t} + \delta n = n_2 |A|^2. \end{cases} \tag{2.1}$$

In what follows, we will write n instead of δn.

2.2 The local Cauchy Problem.

2.2.1 Setting.

We set $\xi = \dfrac{c}{n_0}t - z$ and $A(x, y, z, t) = \bar{A}(\xi, t; x, y)$, $n(x, y, z, t) = \bar{n}(\xi, t; x, y)$. Maxwell-Debye equations (2.1), after this change of variable, read

$$\begin{cases} \dfrac{n_0}{c}\dfrac{\partial \bar{A}}{\partial t} - \dfrac{i}{2k}\nabla_1^2 \bar{A} + i\dfrac{\omega_0}{c}\bar{n}\bar{A} = 0, \\ \tau \dfrac{\partial \bar{n}}{\partial t} + \dfrac{\tau c}{n_0}\dfrac{\partial \bar{n}}{\partial \xi} + \bar{n} = n_2 |\bar{A}|^2. \end{cases} \tag{2.2}$$

Anew we make a change of variable $\bar{n} = \bar{m}e^{-\frac{n_0}{\tau c}\xi}$, yielding the new system :

$$\begin{cases} \frac{n_0}{c}\frac{\partial \bar{A}}{\partial t} - \frac{i}{2k}\nabla_1^2 \bar{A} + i\frac{\omega_0}{c}\bar{m}e^{-\frac{n_0}{\tau c}\xi}\bar{A} = 0, \\ \tau\frac{\partial \bar{m}}{\partial t} + \frac{\tau c}{n_0}\frac{\partial \bar{m}}{\partial \xi} = n_2|\bar{A}|^2 e^{\frac{n_0}{\tau c}\xi}. \end{cases}$$

We study the transport equation with t as evolution variable, and hence, setting $\Xi(t) = \xi_0 + \frac{c}{n_0}t$, where ξ_0 refers to a particular caracteristic,

$$\begin{aligned} \frac{d}{dt}\left(\bar{m}(\Xi(t), t; x, y)\right) &= \left(\frac{\partial \bar{m}}{\partial \xi}\Xi'(t) + \frac{\partial \bar{m}}{\partial t}\right)(\Xi(t), t; x, y), \\ &= \left(\frac{c}{n_0}\frac{\partial \bar{m}}{\partial \xi} + \frac{\partial \bar{m}}{\partial t}\right)(\Xi(t), t; x, y), \\ &= \frac{n_2}{\tau}|\bar{A}\left(\Xi(t), t; x, y\right)|^2 e^{\frac{n_0}{\tau c}\Xi(t)}. \end{aligned}$$

$$\bar{m}(\Xi(t), t; x, y) = \bar{m}(\Xi(t_0), t_0; x, y) + \int_{t_0}^{t} \frac{n_2}{\tau}|\bar{A}\left(\Xi(\zeta), \zeta; x, y\right)|^2 e^{\frac{n_0}{\tau c}\Xi(\zeta)} d\zeta.$$

From now on, we stay on the caracteristic containing the point $(t, \xi) = (0, \xi_0)$, hence $\Xi(t) = \xi_0 + \frac{c}{n_0}t$ and $\Xi(\zeta) = \xi_0 + \frac{c}{n_0}\zeta$.

$$\begin{aligned} \bar{m}(\xi_0 + \frac{c}{n_0}t, t; x, y) &= \bar{m}(\xi_0 + \frac{c}{n_0}t_0, t_0; x, y) \\ &+ \int_{t_0}^{t} \frac{n_2}{\tau}|\bar{A}(\xi_0 + \frac{c}{n_0}\zeta, \zeta; x, y)|^2 e^{\frac{n_0}{\tau c}(\xi_0 + \frac{c}{n_0}\zeta)} d\zeta. \end{aligned}$$

The first variable is useless since we only consider one caracteristic, therefore we set

$$\begin{cases} \bar{A}(\xi_0 + \frac{c}{n_0}t, t; x, y) &= \tilde{A}(t; x, y), \\ \bar{m}(\xi_0 + \frac{c}{n_0}t, t; x, y) &= \tilde{m}(t; x, y), \\ \bar{n}(\xi_0 + \frac{c}{n_0}t, t; x, y) &= \tilde{n}(t; x, y). \end{cases}$$

$$\tilde{m}(t; x, y) = \tilde{m}(t_0; x, y) + \int_{t_0}^{t} \frac{n_2}{\tau}|\tilde{A}(\zeta; x, y)|^2 e^{\frac{n_0}{\tau c}(\xi_0 + \frac{c}{n_0}\zeta)} d\zeta,$$

and accordingly

$$\tilde{n}(t;x,y) = \tilde{n}(t_0;x,y) + \int_{t_0}^{t} \frac{n_2}{\tau} |\tilde{A}(\zeta;x,y)|^2 e^{\frac{\zeta}{\tau}} d\zeta.$$

We combine this result with the Schrödinger equation. We still do not choose a particular t_0 but we suppose it is fixed.

$$\frac{\partial \tilde{A}}{\partial t}(t;x,y) - \frac{ic}{2kn_0}\nabla_1^2 \tilde{A}(t;x,y) +$$
$$+ i\frac{\omega_0}{n_0}\left\{\tilde{n}(t_0;x,y) + \int_{t_0}^{t} \frac{n_2}{\tau}|\tilde{A}(\zeta;x,y)|^2 e^{\frac{\zeta}{\tau}} d\zeta\right\} e^{-\frac{t}{\tau}}\tilde{A}(t;x,y) = 0.$$

We now write the integral formulation of this equation using operator $U(t)$ associated to the linear part $\dfrac{\partial \tilde{A}}{\partial t} - \dfrac{ic}{2kn_0}\nabla_1^2 \tilde{A} = 0$ of the former equation and omitting variables x and y.

$$\tilde{A}(t) = U(t-t_1)\tilde{A}(t_1) - \int_{t_1}^{t} U(t-\theta) i\frac{\omega_0}{n_0}\left\{\tilde{n}(t_0) + \int_{t_0}^{\theta} \frac{n_2}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta}{\tau}} d\zeta\right\} e^{-\frac{\theta}{\tau}}\tilde{A}(\theta) d\theta.$$

Let us fix the initial data, i.e. values for t_0 and t_1.
We arbitrarily set $t_0 = t_1 = 0$. Initial data for $\tilde{A}$ and $\tilde{n}$, are called φ and ν respectively. Thus we get

$$\tilde{A}(t) = U(t)\varphi - \int_{0}^{t} U(t-\theta) i\frac{\omega_0}{n_0}\left\{\nu + \int_{0}^{\theta} \frac{n_2}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta}{\tau}} d\zeta\right\} e^{-\frac{\theta}{\tau}}\tilde{A}(\theta) d\theta. \quad (2.3)$$

Proposition 1 *If A and n belong to $L^\infty(z;0,T;L^2)$, then problems (2.2) and (2.3) are equivalent.*

We try to carry out a fixed point method on formulation (2.3) in order to prove local existence for the Cauchy problem. From now on, we treat $t > 0$ as a time variable and we consider two space variables x and y. Boundary conditions in x and y for A are 0 at $+\infty$ and $-\infty$.
We will perform a fixed point method. To that aim, we set

$$\Phi\tilde{A}(t) = U(t)\varphi - \int_{0}^{t} U(t-\theta) i\frac{\omega_0}{n_0}\left\{\nu + \int_{0}^{\theta} \frac{n_2}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta}{\tau}} d\zeta\right\} e^{-\frac{\theta}{\tau}}\tilde{A}(\theta) d\theta.$$

We want to show that for a certain functionnal space X, $R > 0$ and $0 < \alpha < 1$ and for all $A, B \in B_X(0,R)$, we have $\Phi A \in B_X(0,R)$ and $\|\Phi A - \Phi B\|_X \leq \alpha\|A - B\|_X$.

2.2.2 Estimates on operators.

Let us first set

$$U \star f(t) = \int_0^t U(t-\theta) f(\theta) d\theta,$$

and similarily

$$h \star g(t) = \int_0^t h(t-\theta) g(\theta) d\theta$$

where $h(\theta) = \frac{1}{\tau} e^{-\theta/\tau}$.
With these notations we have :

$$\Phi \tilde{A}(t) = U(t)\varphi - i\frac{\omega_0}{n_0} U \star \left(\nu + n_2 h \star \{|\tilde{A}|^2\}\tilde{A}\right).$$

In this section we state some results about this operator $\star$.
First of all we recall the definition of an admissible pair :
A pair (q,r) is said to be admissible if $\frac{N}{2} - \frac{N}{r} = \frac{2}{q}$ where N is the space dimension and $r \in \left[2, \frac{2N}{N-2}\right)$ ($[2,\infty)$ if $N=2$, $[2,\infty]$ if $N=1$).

Lemma 1 *There exists some $K>0$ such that for all $f \in L^1(0,T;H^s)$,*

$$\|U \star f\|_{L^\infty(0,T;H^s)} \leq K \|f\|_{L^1(0,T;H^s)}.$$

Lemma 2 *If (q,r) and (γ,ρ) are admissible pairs, there exists $K>0$ such that*

$$\|U \star f\|_{L^q(0,T;W^{s,r})} \leq K \|f\|_{L^{\gamma'}(0,T;W^{s,\rho'})},$$

for all $f \in L^{\gamma'}(0,T;W^{s,\rho'})$.

Lemma 3 (Strichartz estimate) *If (q,r) is an admissible pair, there exists a constant C only depending on N and r such that for all $\varphi \in L^2$,*

$$\|U(t)\varphi\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \leq C\|\varphi\|_{L^2}.$$

The proof of the two first lemmas may be found in Ginibre and Velo's article [5], for Strichartz estimate see [9].

Since h belongs to $L^1(0,T)$ with norm $(1-e^{-T/\tau}) < \min(1, \frac{T}{\tau})$, for some positive function g, we have

$$\|h \star g\|_{L^q(0,T;W^{s,p})} \leq \|g\|_{L^q(0,T;W^{s,p})},$$

2.2.3 Existence and Uniqueness for smooth solutions.

The first idea is to seek smooth solutions, i.e. treat the case when X is an algebra, thus we will set $X = L^\infty(0,T;H^s) \cap L^4(0,T;W^{s,4})$ with $s > 1$.
Remark :
If we consider the initial variables, this corresponds to a $L^\infty(\xi; L^\infty(0,T;H^s)) \cap L^\infty(\xi; L^4(0,T;W^{s,4}))$ regularity.

Theorem 1 *i) For all (φ,ν) belonging to $H^s \times H^s$ with $s > 1$, equation (2.3) has a unique solution in $X = L^\infty(0,T;H^s) \cap L^4(0,T;W^{s,4})$ for some small enough T.*
ii) Solutions depend continuously on the initial data, i.e. :
if $\tilde{A} \in L^\infty(0,T;H^s) \cap L^4(0,T;W^{s,4})$ is solution to Maxwell-Debye equations for the initial data (φ,ν), φ_p and ν_p tend respectively to φ and ν in H^s, then for some large enough p, solution $\tilde{A}_p$ to Maxwell-Debye equations associated to initial data φ_p and ν_p tend to $\tilde{A}$ in $L^\infty(0,T;H^s) \cap L^4(0,T;W^{s,4})$.

Proof of i)
Let us set $\Phi\tilde{A}(t) = I + II + III$ with

$$I = U(t)\varphi,$$

$$II = -i\frac{\omega_0}{n_0} U \star \left(\nu e^{-\frac{t}{\tau}}\tilde{A}\right)(t),$$

$$III = -i\frac{\omega_0 n_2}{n_0} U \star \left(h \star \{|\tilde{A}|^2\}\tilde{A}\right)(t).$$

Thank to the above estimates, it is straightforward that for any admissible pair (q,r) where $r \leq 4$:

$$\|I\|_{L^q(0,T;W^{s,r})} = C_r \|\varphi\|_{H^s},$$

where $C_2 = 1$,

$$\|II\|_{L^q(0,T;W^{s,r})} \leq CT\|\nu\|_{H^s}\|\tilde{A}\|_{L^\infty(0,T;H^s)},$$

$$\|III\|_{L^q(0,T;W^{s,r})} \leq CT\|\tilde{A}\|^3_{L^\infty(0,T;H^s)}.$$

Hence

$$\|\Phi\tilde{A}\|_X \leq (1+C_4)\|\varphi\|_{H^s} + CT\|\nu\|_{H^s}\|\tilde{A}\|_X + CT\|\tilde{A}\|^3_X.$$

Let $a = \|\varphi\|_{H^s}$, we set $R = 2(1+C_4)a$. For some small enough time T, we do have

$$\|\Phi\tilde{A}\|_X \leq a + CT\|\nu\|_{H^s}R + CTR^3 \leq R.$$

We next have to verify that Φ is a contraction.

$$(\Phi\tilde{A} - \Phi\tilde{B})(t) = I' + II'$$

where

$$I' = -i\frac{\omega_0}{n_0}U \star \left(\nu e^{-\frac{t}{\tau}}(\tilde{A} - \tilde{B})\right)(t),$$

$$II' = -i\frac{\omega_0 n_2}{n_0}\left[U \star \left(h \star \{|\tilde{A}|^2\}\tilde{A}\right) - U \star \left(h \star \{|\tilde{B}|^2\}\tilde{B}\right)\right](t).$$

The same estimates yield :

$$\|I'\|_{L^q(0,T;W^{s,r})} \leq CT\|\nu\|_{H^s}\|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;H^s)},$$

$$\begin{aligned}\|II'\|_{L^q(0,T;W^{s,r})} \;\; &\leq \;\; CT\left(\|\tilde{A}\|^2_{L^\infty(0,T;H^s)} + \|\tilde{B}\|^2_{L^\infty(0,T;H^s)}\right) \\ &\qquad \times\|\tilde{A} - \tilde{B}\|_{L^\infty(0,T;H^s)}.\end{aligned}$$

Hence

$$\|(\Phi\tilde{A} - \Phi\tilde{B})\|_X \leq CT\left\{\|\nu\|_{H^s} + \|\tilde{A}\|^2_X + \|\tilde{B}\|^2_X\right\}\|\tilde{A} - \tilde{B}\|_X.$$

Keeping the same R but possibly reducing T once more, Φ is contractant in $B_X(0,R)$. We obtain local existence and uniqueness in time for smooth solutions to Maxwell-Debye equations in an integral form. This ends the first step of the proof of theorem 1.

Proof of ii)
The second step is proved in the same way than the contraction.
This completes the proof. ■

Remark : In the former proof, right hand sides only depend on norms in $L^\infty(0,T;H^s)$. Hence we might have done the same analysis in $L^\infty(0,T;H^s)$ only.

2.2.4 Existence and Uniqueness of weaker solutions.

Now let us study weaker solutions, i.e. in $X' = L^\infty(0,T;H^1) \cap L^4(0,T;W^{1,4})$ and $X" = L^4(0,T;L^4) \cap \mathcal{C}(0,T;L^2)$.

Theorem 2 *i) For all (φ, ν) belonging to $H^1 \times H^1$, equation (2.3) has a unique solution in $X' = L^\infty(0,T;H^1) \cap L^4(0,T;W^{1,4})$ for some small enough T.*
ii) For all (φ, ν) belonging $L^2 \times L^2$, equation (2.3) has a unique solution belonging to $X" = L^4(0,T;L^4) \cap \mathcal{C}([0,T];L^2)$ for some small enough T. Moreover $\tilde{A}$ belongs to $L^q(0,T;L^r)$ for every admissible pair (q,r).
iii) Solutions depend continuously on the initial data in an analogous sense to that given by theorem 1.

Proof of i)
Let us show that Φ maps some ball $B_{X'}(0,R)$ in itself.

Le (q,r) be an admissible pair. We obtain the following estimates :

$$\|I\|_{L^q(0,T;W^{1,r})} = C_r \|\varphi\|_{H^1},$$

where $C_2 = 1$,

$$\begin{aligned} \|II\|_{L^q(0,T;L^r)} &\leq C \left\| e^{-\frac{t}{\tau}} \nu \tilde{A}(t) \right\|_{L^2(0,T;L^1)}, \\ &\leq C T^{1/2} \|\nu\|_{L^2} \|\tilde{A}\|_{L^\infty(0,T;L^2)}. \end{aligned}$$

$$\begin{aligned} \|III\|_{L^q(0,T;L^r)} &\leq C \left\| h \star \{|\tilde{A}|^2\} \tilde{A} \right\|_{L^{4/3}(0,T;L^{4/3})}, \\ &\leq C \left\| h \star \{|\tilde{A}|^2\} \right\|_{L^2(0,T;L^2)} \left\| \tilde{A} \right\|_{L^4(0,T;L^4)}, \\ &\leq C \|h\|_{L^1} \||\tilde{A}|^2\|_{L^2(0,T;L^2)} \left\| \tilde{A} \right\|_{L^4(0,T;L^4)}, \\ &\leq C T^{3/4} \|\tilde{A}\|^3_{L^\infty(0,T;H^1)}. \end{aligned}$$

Let us now study the gradients :
Let us set $\nabla_1 II = II_1 + II_2$ with

$$II_1 = -i\frac{\omega_0}{n_0} U \star \left(\nu e^{-\frac{t}{\tau}} \nabla_1 \tilde{A}\right)(t),$$

$$II_2 = -i\frac{\omega_0}{n_0} U \star \left(\nabla_1 \nu e^{-\frac{t}{\tau}} \tilde{A}\right)(t).$$

$$\begin{aligned} \|II_1\|_{L^q(0,T;L^r)} &\leq C \left\| e^{-\frac{t}{\tau}} \nu \nabla_1 \tilde{A} \right\|_{L^2(0,T;L^1)}, \\ &\leq CT^{1/2} \|\nu\|_{L^2} \|\nabla_1 \tilde{A}\|_{L^\infty(0,T;L^2)}. \end{aligned}$$

$$\begin{aligned} \|II_2\|_{L^q(0,T;L^r)} &\leq C \left\| e^{-\frac{t}{\tau}} \nabla_1 \nu \tilde{A} \right\|_{L^2(0,T;L^1)}, \\ &\leq CT^{1/2} \|\nabla_1 \nu\|_{L^2} \|\tilde{A}\|_{L^\infty(0,T;L^2)}. \end{aligned}$$

We may now write $\nabla_1 III = III_1 + III_2 + III_3$ with

$$III_1 = -i\frac{\omega_0 n_2}{n_0} U \star \left(h \star \left\{|\tilde{A}|^2\right\} \nabla_1 \tilde{A}\right)(t),$$

$$III_2 = -i\frac{\omega_0 n_2}{n_0} U \star \left(h \star \left\{\nabla_1 \tilde{A} \tilde{A}^*\right\} \tilde{A}\right)(t),$$

$$III_3 = -i\frac{\omega_0 n_2}{n_0} U \star \left(h \star \left\{\tilde{A} \nabla_1 \tilde{A}^*\right\} \tilde{A}\right)(t).$$

$$\begin{aligned} \|III_1\|_{L^q(0,T;L^r)} &\leq C \|h \star |\tilde{A}|^2\|_{L^{4/3}(0,T;L^4)} \|\nabla_1 \tilde{A}\|_{L^\infty(0,T;L^2)}, \\ &\leq C \|\tilde{A}\|^2_{L^{8/3}(0,T;L^8)} \|\nabla_1 \tilde{A}\|_{L^\infty(0,T;L^2)}, \\ &\leq CT^{3/4} \|\tilde{A}\|^3_{L^\infty(0,T;H^1)}. \end{aligned}$$

We also have :

$$\begin{aligned} \|III_2\|_{L^q(0,T;L^r)} &\leq C \left\| h \star \left(\nabla_1 \tilde{A} \tilde{A}^*\right) \right\|_{L^{4/3}(0,T;L^{8/5})} \|\tilde{A}\|_{L^\infty(0,T;L^8)}, \\ &\leq C \|\nabla_1 \tilde{A}\|_{L^\infty(0,T;L^2)} \|\tilde{A}^*\|_{L^{4/3}(0,T;L^8)} \|\tilde{A}\|_{L^\infty(0,T;L^8)}, \\ &\leq CT^{3/4} \|\tilde{A}\|^3_{L^\infty(0,T;H^1)}. \end{aligned}$$

In the same way
$$\|III_3\|_{L^q(0,T;L^r)} \leq CT^{3/4}\|\tilde{A}\|^3_{L^\infty(0,T;H^1)}.$$
If we set for example $R = 2(1+C_4)\|\varphi\|_{X'}$, and choose some small enough T, Φ maps the ball $B_{X'}(0,R)$ in itself.

The proof for the contraction is done using the same sort of decompositions as in the case of H^s solutions and with the same estimates than above. With a possible new reduction of T, Φ is a contraction from $B_{X'}(0,R)$ in itself. This shows the first part of theorem 2.

Proof of ii)
Most of the useful estimates have been derived in the course of the proof of i). We now follow Cazenave and Weissler's proof [3] for the Schrödinger equation in L^2 with the critical exponent.
Let (q,r) be an admissible pair.
$$\|I\|_{L^q(0,T;L^r)} \leq C_4\|\varphi\|_{L^2},$$
$$\|II\|_{L^q(0,T;L^r)} \leq C'T^{1/2}\|\nu\|_{L^2}\|\tilde{A}\|_{L^4(0,T;L^4)},$$
$$\|III\|_{L^q(0,T;L^r)} \leq C\min\left(\frac{T}{\tau},1\right)\|\tilde{A}\|^3_{L^4(0,T;L^4)}.$$
We first choose $r = q = 4$ and with the same sort of estimates for the differences, we find out that Φ is a contraction in $B_{L^4(0,T;L^4)}(0,R)$ for some small enough T and R such that $C_4\|\varphi\|_{L^2}$ be lower than $\dfrac{R}{2}$.
This yields existence and uniqueness of a solution in this space. Now considering any admissible pair (q,r), we find that $\tilde{A}$ belongs to $\mathcal{C}([0,T];L^2)$ and $L^q(0,T;L^r)$.

Proof of iii)
The proof for the continuity with respect to the initial data is similar to that of theorem 1 using the same type of estimates than above. ■

Remark : In the proof of theorem 2, right hand sides only depend on norms in $L^\infty(0,T;H^1)$. Hence we might have done the same analysis in $L^\infty(0,T;H^1)$ only. We use $L^\infty(0,T;H^1) \cap L^4(0,T;W^{1,4})$ because this space is needed for global existence proofs.

2.2.5 Regularity result.

In what follows, we show that the existence time of the solution to Maxwell-Debye equations for some given initial data is the same in all the spaces H^s, $s > 1$. This result is given by

Theorem 3 *Let* $(\varphi, \nu) \in H^{1+\epsilon} \times H^{1+\epsilon}$, $\epsilon > 0$, *and* $\tilde{A}$ *be the maximal solution to Maxwell-Debye equations in* $H^{1+\epsilon}$. *Let* $T_{1+\epsilon}$ *be its existence time. Let us moreover assume that* $(\varphi, \nu) \in H^s \times H^s$ *with* $s > 1 + \epsilon$, *then* $\tilde{A}$ *is solution to Maxwell-Debye equations in* $L^\infty(0, T_{1+\epsilon}; H^s)$.

Proof :
To prove this, we consider the integro-differential form of Maxwell-Debye equations :

$$\frac{\partial \tilde{A}}{\partial t} - \frac{ic}{2kn_0}\nabla_1^2 \tilde{A} + i\frac{\omega_0}{n_0}\left\{\nu + \int_0^t \frac{n_2}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta}{\tau}} d\zeta\right\} e^{-\frac{t}{\tau}}\tilde{A}(t) = 0.$$

Let J^s be the operator $(1 - \nabla_1^2)^s$. We denote by $\langle \cdot, \cdot \rangle$, the scalar product in $L^2(dx, dy)$.

$$\text{Re } \langle \frac{\partial \tilde{A}}{\partial t}, J^s \tilde{A}\rangle = \frac{1}{2}\frac{d}{dt}\|\tilde{A}(t)\|^2_{H^s},$$
$$\text{Re } \langle i\nabla_1^2 \tilde{A}, J^s \tilde{A}\rangle = 0,$$

$$\begin{aligned}
\left|\langle \nu e^{-\frac{t}{\tau}}\tilde{A}(t), J^s\tilde{A}\rangle\right| &\leq \langle J^{s/2}\nu e^{-\frac{t}{\tau}}\tilde{A}(t), J^{s/2}\tilde{A}\rangle, \\
&\leq \|\tilde{A}(t)\|_{H^s}\|\nu\tilde{A}(t)\|_{H^s}, \\
&\leq C\|\tilde{A}(t)\|_{H^s}\left(\|\nu\|_{L^\infty}\|\tilde{A}(t)\|_{H^s} + \|\nu\|_{H^s}\|\tilde{A}(t)\|_{L^\infty}\right),
\end{aligned}$$

$$\begin{aligned}
\left|\langle \int_0^t |\tilde{A}(\zeta)|^2\tilde{A}(t)e^{\frac{(\zeta-t)}{\tau}}d\zeta, J^s\tilde{A}\rangle\right| &\leq \langle J^{s/2}\int_0^t |\tilde{A}(\zeta)|^2\tilde{A}(t)e^{\frac{(\zeta-t)}{\tau}}d\zeta, J^{s/2}\tilde{A}\rangle, \\
&\leq \|\tilde{A}(t)\|_{H^s}\int_0^t \||\tilde{A}(\zeta)|^2\tilde{A}(t)\|_{H^s}d\zeta, \\
&\leq C\|\tilde{A}(t)\|_{H^s}\int_0^t (\|\tilde{A}(\zeta)\|_{H^s}\|\tilde{A}(\zeta)\|_{L^\infty}\|\tilde{A}(t)\|_{L^\infty} \\
&\qquad +\|\tilde{A}(\zeta)\|^2_{L^\infty}\|\tilde{A}(t)\|_{H^s})d\zeta,
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|\tilde{A}(t)\|_{H^s}\|\tilde{A}(t)\|_{L^\infty}\int_0^t \|\tilde{A}(\zeta)\|_{H^s}\|\tilde{A}(\zeta)\|_{L^\infty}d\zeta \\
&\quad +C\|\tilde{A}(t)\|_{H^s}^2\int_0^t \|\tilde{A}(\zeta)\|_{L^\infty}^2 d\zeta.
\end{aligned}
$$

We assumed that $\tilde{A} \in L^\infty(0,T;H^{1+\epsilon})$ (i.e. $T < T_{1+\epsilon},$) and that $\|\nu\|_{H^s}$ and $\|\varphi\|_{H^s}$ are finite for $s > 1+\epsilon$. Using the fact that $H^{1+\epsilon} \hookrightarrow L^\infty$, and setting $y = \displaystyle\int_0^t \|\tilde{A}(\zeta)\|_{H^s}d\zeta$, we obtain the differential inequality :

$$
y'y'' \leq C(y'^2 + yy').
$$

As $y' = \|\tilde{A}(t)\|_{H^s} > 0$, we may divide by this quantity and set $z = y + y'$. z is solution to the equation $z' \leq (C+1)z$ with $z(0) = \|\varphi\|_{H^s}$, therefore

$$
\|\tilde{A}(t)\|_{H^s} = y' \leq z' \leq \|\varphi\|_{H^s}e^{(C+1)t}.
$$

■

2.3 Limit as the delay tends to 0.

2.3.1 Existence of solution on a time interval independent of τ.

The former estimates are uniform with respect to τ (as it tends to 0). Indeed for the study in H^s, we have

$$
\|\Phi\tilde{A}\|_X \leq \|\varphi\|_{H^s} + CT\|\nu\|_{H^s}\|\tilde{A}\|_X + CT\|\tilde{A}\|_X^3.
$$

$$
\|(\Phi\tilde{A} - \Phi\tilde{B})\|_X \leq CT\left\{\|\nu\|_{H^s} + \|\tilde{A}\|_X^2 + \|\tilde{B}\|_X^2\right\}\|\tilde{A} - \tilde{B}\|_X.
$$

We notice that the situation is the same (with other powers for T) in the case of H^1 estimates. Therefore we may state that the solutions to Maxwell-Debye equations (in the two former functional contexts) exist on a time interval $[0,T]$ which does not depend on τ. With this result, we may study the limit of the solutions to Maxwell-Debye equations as τ tends to 0. Since equations formally tend to the cubic Schrödinger equation

$$
\frac{\partial \tilde{A}}{\partial t} - \frac{ic}{2kn_0}\nabla_1^2\tilde{A} + i\frac{\omega_0 n_2}{n_0}|\tilde{A}|^2\tilde{A} = 0, \qquad (2.4)
$$

we hope that the solutions will tend to the solution to this equation.

2.3.2 Passing to the limit for strong solutions.

Theorem 4 *We assume that the initial data (for $\tilde{A}$ and $\tilde{n}$) are uniformly bounded in $X = L^\infty(0,T;H^s)$, $s > 3$, and that as τ tends to 0, the initial data φ strongly tend to ψ in H^s. Let A be the solution to the cubic Schrödinger equation associated to these initial data ψ. Then, as τ tends to 0, the sequence of $\tilde{A}$ strongly tends to A in X.*

To show this theorem, we will use as main ingredient Ascoli-Arzela's theorem.
The assumptions on the initial data in theorem 4 ensure that the sequence of solutions is uniformly bounded in X. Moreover

$$\frac{\partial \tilde{A}}{\partial t}(t) - \frac{ic}{2kn_0}\nabla_1^2 \tilde{A}(t) + \\ +i\frac{\omega_0}{n_0}\left\{\nu e^{-t/\tau}\tilde{A}(t) + n_2\left(h \star |\tilde{A}|^2\right)\right\}\tilde{A}(t) = 0.$$

We consider the case when H^{s-2} is an algebra (for more simplicity in the computations), i.e. $s > 3$. Then

$$\begin{aligned}
\left\|\frac{\partial \tilde{A}}{\partial t}\right\|_{H^{s-2}} &\leq \frac{c}{2kn_0}\left\|\tilde{A}(t)\right\|_{H^s} \\
&\quad + \frac{\omega_0}{n_0}\|\nu\|_{H^{s-2}} e^{-\frac{t}{\tau}}\left\|\tilde{A}(t)\right\|_{H^{s-2}} \\
&\quad + \frac{\omega_0 n_2}{n_0}\|h \star |\tilde{A}|^2\|_{H^{s-2}}\left\|\tilde{A}(t)\right\|_{H^{s-2}} \\
&\leq \frac{c}{2kn_0}\left\|\tilde{A}(t)\right\|_{H^s} \\
&\quad + \frac{\omega_0}{n_0}\left(\|\nu\|_{H^{s-2}} + n_2\|\tilde{A}\|_X^2\right)\|\tilde{A}(t)\|_{H^{s-2}}.
\end{aligned}$$

For some small enough T, this is bounded uniformly in τ in X. By Ascoli-Arzela's theorem, we may state that there exists a sub-sequence $\tilde{A}$ which tends in $\mathcal{C}(0,T;H^{s-\epsilon})$ for all $\epsilon > 0$ to a function A as τ tends to 0.
There remains to check that this limit is solution to the cubic Schrödinger equation (2.4).
We easily notice that $\nu e^{-\frac{t}{\tau}}\tilde{A}(t)$ tends to 0 as τ tends to 0. The only term which sets us a problem is the nonlinear one. We also want to show that

$\int_0^t \frac{1}{\tau}|\tilde{A}(\zeta;x,y)|^2 e^{\frac{\zeta-t}{\tau}}d\zeta$ tends in a certain sense to $|A(t)|^2$.

$$\int_0^t \frac{1}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta-t}{\tau}}d\zeta - |A(t)|^2$$
$$= \int_0^{t-\eta} \frac{1}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta-t}{\tau}}d\zeta + \int_{t-\eta}^t \frac{1}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta-t}{\tau}}d\zeta - |A(t)|^2,$$
$$= \int_0^{t-\eta} \frac{1}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta-t}{\tau}}d\zeta + \int_{t-\eta}^t \frac{1}{\tau}(|\tilde{A}(\zeta)|^2 - |A(\zeta)|^2) e^{\frac{\zeta-t}{\tau}}d\zeta$$
$$+ \int_{t-\eta}^t \frac{1}{\tau}(|A(\zeta)|^2 - |A(t)|^2) e^{\frac{\zeta-t}{\tau}}d\zeta - e^{-\frac{\eta}{\tau}}|A(t)|^2.$$

We very easily estimate each term of this sum in the H^σ norm where σ is lower than s.

By the convergence result, for all α there exists some τ_0 such that for all $\tau < \tau_0$,

$$\left\| \int_{t-\eta}^t \frac{1}{\tau}(|\tilde{A}(\zeta)|^2 - |A(\zeta)|^2) e^{\frac{\zeta-t}{\tau}}d\zeta \right\|_{H^\sigma} \leq \alpha.$$

By continuity and hence uniform continuity on $[0,T]$ for A

$$\left\| |A(\zeta)|^2 - |A(t)|^2 \right\|_{H^\sigma} \leq \alpha$$

as soon as $\zeta - t < \eta$ and therefore

$$\left\| \int_{t-\eta}^t \frac{1}{\tau}(|A(\zeta)|^2 - |A(t)|^2) e^{\frac{\zeta-t}{\tau}}d\zeta \right\|_{H^\sigma} \leq \epsilon.$$

For some given ϵ this fixes η. For this η,

$$\left\| \int_0^{t-\eta} \frac{1}{\tau}|\tilde{A}(\zeta)|^2 e^{\frac{\zeta-t}{\tau}}d\zeta \right\|_{H^\sigma} \leq \left\| \tilde{A} \right\|_X^2 \left(e^{-\frac{\eta}{\tau}} - e^{-\frac{t}{\tau}} \right),$$

which tends uniformly to 0 on $[0,T]$. And last $\left\| e^{-\frac{\eta}{\tau}}|A(t)|^2 \right\|_{H^\sigma}$ tends to 0 as τ tends to 0.

We notice that all these results are uniform with respect to the time, hence we obtain the strong convergence in $L^\infty(0,T;H^{s-\epsilon})$.

Thus we did prove that a sub-sequence (and consequently the whole sequence) of $\tilde{A}$ converges in $L^\infty(0,T;H^{s-\epsilon})$ to the solution to the cubic Schrödinger equation.

2.3.3 Passing to the limit for weak solutions.

We proceed in a similar way to the smooth solutions case. We may not use the algebra structure any longer, but estimates are still easy to obtain. Indeed :

$$\begin{aligned}
\left\|\frac{\partial\tilde{A}}{\partial t}\right\|_{L^\infty(0,T;H^{-1})} \leq\ & \frac{C}{2kn_0}\left\|\tilde{A}(t)\right\|_{L^\infty(0,T;H^1)} + \frac{\omega_0}{n_0}\left\|\nu e^{-\frac{t}{\tau}}\tilde{A}(t)\right\|_{L^\infty(0,T;H^{-1})}\\
&+\frac{\omega_0 n_2}{n_0}\left\|\left(h\star|\tilde{A}(\zeta)|^2\right)\tilde{A}(t)\right\|_{L^\infty(0,T;H^{-1})},\\
\leq\ & \frac{c}{2kn_0}\left\|\tilde{A}(t)\right\|_{L^\infty(0,T;H^1)} + \frac{\omega_0}{n_0}\left\|\nu e^{-\frac{t}{\tau}}\tilde{A}(t)\right\|_{L^\infty(0,T;L^{3/4})}\\
&+\frac{\omega_0 n_2}{n_0}\left\|\left(h\star|\tilde{A}(\zeta)|^2\right)\tilde{A}(t)\right\|_{L^\infty(0,T;L^{3/4})},\\
\leq\ & \frac{c}{2kn_0}\left\|\tilde{A}(t)\right\|_{L^\infty(0,T;H^1)} + \frac{\omega_0}{n_0}\|\nu\|_{L^{3/2}}\left\|\tilde{A}(t)\right\|_{L^\infty(0,T;L^{3/2})}\\
&+\frac{\omega_0 n_2}{n_0}\left\|h\star|\tilde{A}(\zeta)|^2\right\|_{L^\infty(0,T;L^2)}\left\|\tilde{A}(t)\right\|_{L^\infty(0,T;L^4)},\\
\leq\ & \frac{c}{2kn_0}\left\|\tilde{A}(t)\right\|_{L^\infty(0,T;H^1)} + \frac{\omega_0}{n_0}\|\nu\|_{H^1}\left\|\tilde{A}(t)\right\|_{L^\infty(0,T;H^1)}\\
&+C\frac{\omega_0 n_2}{n_0}T^{1/2}\left\|\tilde{A}(t)\right\|^3_{L^\infty(0,T;H^1)}.
\end{aligned}$$

The rest of the argument is analogous to the former one. We may state the following result :

Theorem 5 *We assume the initial data (for $\tilde{A}$ and $\tilde{n}$) to be uniformly bounded in $X = L^\infty(0,T;H^1)$, and that as τ tends to 0, the initial data φ strongly tends to ψ in H^1. Let A be the solution to the cubic Schrödinger equation associated to the initial data ψ. Then, as τ tends to 0, the sequence $\tilde{A}$ strongly tends to A in X.*

3 Maxwell-Bloch equations.

3.1 Modelization.

We will now write Maxwell-Bloch equations which describe the interactions of an unidirectional electromagnetic wave with a medium of gas of two-level atoms. As we neglect the Doppler effect, polarisation reads

$$\vec{P} = n_a(\vec{p}_{12}\rho_{12} + \vec{p}_{21}\rho_{21}).$$

We moreover suppose that the field is polarized in a unique direction which is perpendicular to its propagation direction z. Without any loss of generality, we may assume that the direction of the dipolar matrix element is parallel to that of the electric field. Hence we have

$$\vec{E} = \vec{A}(x, y, z, t)e^{i(\omega/c)z}e^{-i\omega_c t} + c.c.$$

(where *c.c.* stands for the complex conjugate) and

$$\vec{P} = \vec{L}(x, y, z, t)e^{i(\omega/c)z}e^{-i\omega_c t} + c.c.$$

Moreover we make the slowly varying envelope approximation. Then equation (1.3) becomes

$$\frac{\partial \vec{A}}{\partial z} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} - i\frac{c}{2\omega}\nabla_1^2\vec{A} + \frac{\kappa}{c}\vec{A} = \frac{i\omega}{2\epsilon_0 c}\vec{L}$$

where κ describes losses due for example to mirrors.
Let us denote by $\hat{e}$ the polarization direction of the field (i.e. $\vec{A}(x, y, z, t) = \hat{e}A(x, y, z, t)$) and p the modulus of $\vec{p}_{12}$. Then we have

$$n_a p\rho_{12} = L(x, y, z, t)e^{i(\omega/c)z - i\omega t}.$$

Writting Bloch's equations for ρ_{12} and $\rho_{22} - \rho_{11} = \dfrac{N}{n_a}$ (we neglect terms containing second harmonics $e^{\pm 2i\omega t}$), assuming that $\gamma_{11} = \gamma_{22}$ and setting

$\omega_{12} = \omega_1 - \omega_2 > 0$, we get

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c}A = \dfrac{i\omega}{2\epsilon_0 c}L, \\ \\ \dfrac{\partial L}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar}AN, \\ \\ \dfrac{\partial N}{\partial t} + \gamma_{11}N = \dfrac{2i}{\hbar}(A^*L - AL^*). \end{cases}$$

We have to pump energy to the medium in order to keep it active, i.e. we force a part of the atoms to be in an excited state. The simulation of this contribution is given by the constant term $\gamma_{11}N_0$ in the equation governing the inversion number N. Maxwell-Bloch equations governing an unidirectional wave, polarized in a single direction, in a medium constituted by two-level atoms are also

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c}A = \dfrac{i\omega}{2\epsilon_0 c}L, \\ \\ \dfrac{\partial L}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar}AN, \\ \\ \dfrac{\partial N}{\partial t} + \gamma_{11}(N - N_0) = \dfrac{2i}{\hbar}(A^*L - AL^*). \end{cases} \tag{3.1}$$

3.2 Study of the quasi-steady state equation.

Neglecting time variations for L and M (adiabatic approximation) we obtain new equations :

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} - i\dfrac{c}{2\omega}\nabla_1^2 A + \dfrac{\kappa}{c}A = \dfrac{i\omega}{2\epsilon_0 c}L, \\ \\ (\gamma_{12} + i(\omega_{12} - \omega))L = \dfrac{ip^2}{\hbar}AN, \\ \\ \gamma_{11}(N - N_0) = \dfrac{2i}{\hbar}(A^*L - AL^*). \end{cases}$$

Carrying out substitutions, we get

$$\frac{\partial A}{\partial z} + \frac{1}{c}\frac{\partial A}{\partial t} - i\frac{c}{2\omega}\nabla_1^2 A + \frac{\kappa}{c}A = \frac{i\omega}{2\epsilon_0 c}L \tag{3.2}$$

where

$$L = \frac{ip^2}{\hbar}\frac{(\gamma_{12} - i(\omega_{12} - \omega))N_0 A}{\gamma_{12}^2 + (\omega_{12} - \omega)^2 + \frac{4p^2\gamma_{12}}{\hbar^2\gamma_{11}}|A|^2}. \tag{3.3}$$

If we set $\xi = ct - z$ and $A(x, y, z, t) = \bar{A}(\xi, t; x, y)$ and analogous notations for $\bar{L}$ and $\bar{N}$, we obtain

$$\frac{\partial \bar{A}}{\partial t} - i\frac{c^2}{2\omega}\nabla_1^2 \bar{A} + \kappa\bar{A} = \frac{i\omega}{2\epsilon_0}\bar{L}.$$

We easily notice that

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy \quad &+ \quad \kappa\int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy \\ &= \quad -\frac{\omega p^2}{2\epsilon_0\hbar}\int_{\mathbb{R}^2} \frac{\gamma_{12}N_0|\bar{A}(t)|^2}{\gamma_{12}^2 + (\omega_{12} - \omega)^2 + \frac{4p^2\gamma_{12}}{\hbar^2\gamma_{11}}|\bar{A}(t)|^2} dxdy. \end{aligned} \tag{3.4}$$

Hence there exists a constant D such that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy \leq D\int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy.$$

The second member has the same sign than $-N_0$.
In the case when $N_0 \geq 0$, we may use 0 as upper bound and set $D = -2\kappa < 0$.
In the case when $N_0 < 0$, we use

$$-\frac{\omega p^2}{2\epsilon_0\hbar}\frac{\gamma_{12}N_0}{\gamma_{12}^2 + (\omega_{12} - \omega)^2}\int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy$$

as upper bound and set $D = -2\kappa - \dfrac{\omega p^2}{\epsilon_0\hbar} \cdot \dfrac{\gamma_{12}N_0}{\gamma_{12}^2 + (\omega_{12} - \omega)^2}$, which is nonpositive if

$$\kappa > \frac{\omega p^2}{2\epsilon_0\hbar} \cdot \frac{-\gamma_{12}N_0}{\gamma_{12}^2 + (\omega_{12} - \omega)^2}.$$

By Gronwall's lemma

$$\int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy \leq \left(\int_{\mathbb{R}^2} |\bar{A}(0)|^2 dxdy \right) e^{Dt}. \qquad (3.5)$$

On a finite time interval, the L^2 norm remains bounded. This bound is independent of time and tends to 0 as time tends to $+\infty$ in all the cases when we may choose $D < 0$.
We clearly have local existence and uniqueness of a solution in L^2 and (3.5) ensures that this result is in fact a global one.
To establish a theory in H^1, it suffices to notice that

$$\nabla_1 L = \frac{CN_0 \nabla_1 \bar{A}(\alpha + \beta|\bar{A}|^2) + CN_0 A(\alpha + \beta[\bar{A}\nabla_1 \bar{A}^* + \bar{A}^* \nabla_1 \bar{A}])}{(\alpha + \beta|\bar{A}|^2)^2}$$

where $C = \dfrac{ip^2}{\hbar}(\gamma_{12} - i(\omega_{12} - \omega))$, $\alpha = \gamma_{12}^2 + (\omega_{12} - \omega)^2$ and $\beta = \dfrac{4p^2\gamma_{12}}{\hbar^2\gamma_{11}}$.
Estimating in the same way than in the course of the study of Maxwell-Debye equations in H^1, we obtain the local existence and uniqueness of a solution to this quasi-steady state equation in H^1.
An other possible estimate is

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(\gamma_{12}^2 + (\omega_{12} - \omega)^2) \int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy + \frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}} \int_{\mathbb{R}^2} |\bar{A}(t)|^4 dxdy\right]$$
$$+\left[\kappa(\gamma_{12}^2 + (\omega_{12} - \omega)^2) + \frac{\omega p^2 \gamma_{12}}{2\epsilon_0 \hbar} N_0\right] \int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy$$
$$+4\kappa \frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}} \int_{\mathbb{R}^2} |\bar{A}(t)|^4 dxdy = 0.$$

If $D < 0$, the quantity

$$\kappa(\gamma_{12}^2 + (\omega_{12} - \omega)^2) + \frac{\omega p^2 \gamma_{12}}{2\epsilon_0 \hbar} N_0$$

is positive. $4\kappa \dfrac{p^2}{\hbar^2}\dfrac{\gamma_{12}}{\gamma_{11}}$ is also always positive. Then there exists a constant C such that

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(\gamma_{12}^2 + (\omega_{12} - \omega)^2) \int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy + \frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}} \int_{\mathbb{R}^2} |\bar{A}(t)|^4 dxdy\right]$$
$$+C\left[\frac{1}{2}(\gamma_{12}^2 + (\omega_{12} - \omega)^2) \int_{\mathbb{R}^2} |\bar{A}(t)|^2 dxdy + \frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}} \int_{\mathbb{R}^2} |\bar{A}(t)|^4 dxdy\right] \leq 0,$$

and by Gronwall's lemma

$$\left[\frac{1}{2}(\gamma_{12}^2+(\omega_{12}-\omega)^2)\int_{\mathbb{R}^2}|\bar{A}(t)|^2dxdy+\frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}}\int_{\mathbb{R}^2}|\bar{A}(t)|^4dxdy\right]$$

$$\leq\left[\frac{1}{2}(\gamma_{12}^2+(\omega_{12}-\omega)^2)\int_{\mathbb{R}^2}|\bar{A}(0)|^2dxdy+\frac{p^2}{\hbar^2}\frac{\gamma_{12}}{\gamma_{11}}\int_{\mathbb{R}^2}|\bar{A}(0)|^4dxdy\right]e^{-Ct}.$$

Hence L^2 and L^4 norms of $A(t)$ decrease with time.
Moreover we notice that $\|\bar{L}(t)\|_{L^2}\leq\dfrac{p^2}{\hbar(\gamma_{12}^2+(\omega_{12}-\omega)^2)^{1/2}}|N_0|\|\bar{A}(t)\|_{L^2}$.

Theorem 6 *The Cauchy problem is globaly well-posed in L^2 and in H^1 for the adiabatic approximation of Maxwell-Bloch equations. Moreover, for certain values of the parameters $\left(\kappa>\dfrac{\omega p^2}{2\epsilon_0\hbar}\cdot\dfrac{-\gamma_{12}N_0}{\gamma_{12}^2+(\omega_{12}-\omega)^2}\right)$, L^2 norms of $\bar{A}$ and $\bar{L}$ tend to 0 when t tends to $+\infty$.*

Remark : This damping is clearly due to the positive coefficients κ and γ_{12}.

3.3 The local Cauchy Problem.

3.3.1 Setting.

Let us consider again Maxwell-Bloch equations in the form :

$$\begin{cases}\dfrac{\partial A}{\partial z}+\dfrac{1}{c}\dfrac{\partial A}{\partial t}-i\dfrac{c}{2\omega}\nabla_1^2A+\dfrac{\kappa}{c}A=\dfrac{i\omega}{2\epsilon_0c}L,\\[2ex] \dfrac{\partial L}{\partial t}+(\gamma_{12}+i(\omega_{12}-\omega))L=\dfrac{ip^2}{\hbar}AN,\\[2ex] \dfrac{\partial N}{\partial t}+\gamma_{11}(N-N_0)=\dfrac{2i}{\hbar}(A^*L-AL^*).\end{cases}$$

Maxwell-Bloch equations, rewritten after the change of variables $\xi=ct-z$ and with the same notations than for the quasi-steady state equation, are in the form :

$$\left\{\begin{array}{l}\dfrac{\partial \bar{A}}{\partial t} - i\dfrac{c^2}{2\omega}\nabla_1^2\bar{A} + \kappa\bar{A} = \dfrac{i\omega}{2\epsilon_0}\bar{L}, \\[2ex] c\dfrac{\partial \bar{L}}{\partial \xi} + \dfrac{\partial \bar{L}}{\partial t} + (\gamma_{12} + i(\omega_{12} - \omega))\bar{L} = \dfrac{ip^2}{\hbar}\bar{A}\bar{N}, \\[2ex] c\dfrac{\partial \bar{N}}{\partial \xi} + \dfrac{\partial \bar{N}}{\partial t} + \gamma_{11}(\bar{N} - N_0) = \dfrac{2i}{\hbar}(\bar{A}^*\bar{L} - \bar{A}\bar{L}^*).\end{array}\right. \tag{3.6}$$

We set $\bar{M} = \bar{N} - N_0$ hence :

$$\left\{\begin{array}{l}\dfrac{\partial \bar{A}}{\partial t} - i\dfrac{c^2}{2\omega}\nabla_1^2\bar{A} + \kappa\bar{A} = \dfrac{i\omega}{2\epsilon_0}\bar{L}, \\[2ex] \dfrac{\partial \bar{L}}{\partial t} + c\dfrac{\partial \bar{L}}{\partial \xi} + (\gamma_{12} + i(\omega_{12} - \omega))\bar{L} = \dfrac{ip^2}{\hbar}(\bar{A}N_0 + \bar{A}\bar{M}), \\[2ex] \dfrac{\partial \bar{M}}{\partial t} + c\dfrac{\partial \bar{M}}{\partial \xi} + \gamma_{11}\bar{M} = \dfrac{2i}{\hbar}(\bar{A}^*\bar{L} - \bar{A}\bar{L}^*).\end{array}\right.$$

We write this in an integral form using the operator U associated to the linear equation $\dfrac{\partial A}{\partial t} - i\dfrac{c^2}{2\omega}\nabla_1^2 A = 0$ and considering as initial time $t_0 = 0$. Initial data for $\bar{A}$, $\bar{L}$ and $\bar{M}$ are respectively called φ, λ and μ.

$$\left\{\begin{array}{rcl}\bar{A}(\xi,t;x,y) & = & U(t)\varphi(\xi;x,y) \\ & + & \displaystyle\int_0^t U(t-\theta)\left[-\kappa\bar{A} + \frac{i\omega}{2\epsilon_0}\bar{L}\right](\xi,\theta;x,y)d\theta, \\[3ex] \bar{L}(\xi,t;x,y) & = & \lambda(\xi - ct;x,y) \\ & + & \displaystyle\int_0^t \left[-(\gamma_{12} + i(\omega_{12} - \omega))\bar{L} + \frac{ip^2}{c\hbar}(\bar{A}N_0 + \bar{A}\bar{M})\right] \\ & & \qquad (\xi - c(t-\theta),\theta;x,y)d\theta, \\[3ex] \bar{M}(\xi,t;x,y) & = & \mu(\xi - ct;x,y) \\ & + & \displaystyle\int_0^t \left[-\gamma_{11}\bar{M} + \frac{2i}{\hbar}(\bar{A}^*\bar{L} - \bar{A}\bar{L}^*)\right](\xi - c(t-\theta),\theta;x,y)d\theta.\end{array}\right. \tag{3.7}$$

Proposition 2 *If $\bar{A}$, $\bar{L}$ and $\bar{M}$ belong to $L^\infty(\xi; 0, T; L^2)$ then problems (3.6) and (3.7) are equivalent.*

We want to use a fixed point method, therefore we set

$$\left\{\begin{array}{rcl}
\Phi_{\bar{A}}(\xi, t; x, y) & = & U(t)\varphi(\xi; x, y) \\
& + & \displaystyle\int_0^t U(t-\theta)\left[-\kappa\bar{A} + \frac{i\omega}{2\epsilon_0}\bar{L}\right](\xi, \theta; x, y)d\theta, \\
\Phi_{\bar{L}}(\xi, t; x, y) & = & \lambda(\xi - ct; x, y) \\
& + & \displaystyle\int_0^t \left[-(\gamma_{12} + i(\omega_{12} - \omega))\bar{L} + \frac{ip^2}{c\hbar}(\bar{A}N_0 + \bar{A}\bar{M})\right] \\
& & \qquad (\xi - c(t-\theta), \theta; x, y)d\theta, \\
\Phi_{\bar{M}}(\xi, t; x, y) & = & \mu(\xi - ct; x, y) \\
& + & \displaystyle\int_0^t \left[-\gamma_{11}\bar{M} + \frac{2i}{\hbar}(\bar{A}^*\bar{L} - \bar{A}\bar{L}^*)\right](\xi - c(t-\theta), \theta; x, y)d\theta.
\end{array}\right.$$

3.3.2 Existence and uniqueness of smooth solutions.

We will seek a solution belonging to $(L^\infty(\xi; L^\infty(0, T; H^s(x, y))))^3 =: X^3$ for $s > 1$.

Theorem 7 *i) For all $(\varphi, \lambda, \mu) \in L^\infty(\xi; H^s) \times L^\infty(\xi; H^s) \times L^\infty(\xi; H^s)$, equation (3.7) has a unique solution in $X^3 = (L^\infty(\xi, 0, T; H^s))^3$ for a small enough T .*
ii) The solutions depend continuously on the initial data in a similar sense to theorem 1.

Proof :

$$\begin{array}{rcl}
\|\Phi_{\bar{A}}(\xi)\|_{L^\infty(0,T;H^s(x,y))} & \leq & \|U(t)\varphi(\xi)\|_{L^\infty(0,T;H^s(x,y))} \\
& + & \left\|\displaystyle\int_0^t U(t-\theta)\left[-\kappa\bar{A} + \frac{i\omega}{2\epsilon_0}\bar{L}\right](\xi; \theta)d\theta\right\|_{L^\infty(0,T;H^s(x,y))}, \\
& \leq & \|\varphi(\xi)\|_{H^s(x,y)} \\
& + & K\left\|-\kappa\bar{A}(\xi) + \dfrac{i\omega}{2\epsilon_0}\bar{L}(\xi)\right\|_{L^1(0,T;H^s(x,y))},
\end{array}$$

$$
\begin{aligned}
&\leq \|\varphi(\xi)\|_{H^s(x,y)} \\
&+ KT\left(\left\|\bar{A}(\xi)\right\|_{L^\infty(0,T;H^s(x,y))} + \left\|\bar{L}(\xi)\right\|_{L^\infty(0,T;H^s(x,y))}\right).
\end{aligned}
$$

Hence

$$
\|\Phi_{\bar{A}}\|_X \leq \|\varphi\|_{\bar{L}^\infty(\xi;H^s(x,y))} + KT\left(\left\|\bar{A}\right\|_X + \left\|\bar{L}\right\|_X\right).
$$

$$
\begin{aligned}
\|\Phi_{\bar{L}}\|_X \leq & \|\lambda\|_{L^\infty(\xi;H^s(x,y))} \\
& + T\left\|-(\gamma_{12} + i(\omega_{12} - \omega))\bar{L} + \frac{ip^2}{\hbar}(\bar{A}N_0 + \bar{A}\bar{M})(\xi - c(t-\theta), \theta)\right\|_X, \\
\leq & \|\lambda\|_{\bar{L}^\infty(\xi;H^s(x,y))} + KT\left(\left\|\bar{L}\right\|_X + \left\|\bar{A}\right\|_X N_0 + \left\|\bar{A}\right\|_X \left\|\bar{M}\right\|_X\right).
\end{aligned}
$$

In the same way

$$
\|\Phi_{\bar{M}}\|_X \leq \|\mu\|_{L^\infty(\xi;H^s(x,y))} + KT\left(\left\|\bar{M}\right\|_X + \left\|\bar{A}\right\|_X \left\|\bar{L}\right\|_X\right).
$$

Hence setting

$$
\frac{R}{2} = \sup\left(\|\varphi\|_{L^\infty(\xi;H^s(x,y))}, \|\lambda\|_{L^\infty(\xi;H^s(x,y))}, \|\mu\|_{L^\infty(\xi;H^s(x,y))}\right),
$$

Φ maps the ball $B_{X^3}(0,R)$ in itself for a small enough T.
Concerning the contraction, we consider two solutions $(\bar{A}_1, \bar{L}_1, \bar{M}_1)$ and $(\bar{A}_2, \bar{L}_2, \bar{M}_2)$ to the system with the same initial data at time $t = 0$, we obtain (omitting variables x and y)

$$
\begin{cases}
(\Phi_{\bar{A}_1} - \Phi_{\bar{A}_2})(\xi;t) &= \displaystyle\int_0^t U(t-\theta)\left[-\kappa(\bar{A}_1-\bar{A}_2)+\frac{i\omega}{2\epsilon_0}(\bar{L}_1-\bar{L}_2)\right](\xi,\theta)d\theta, \\
(\Phi_{\bar{L}_1} - \Phi_{\bar{L}_2})(\xi;t) &= \displaystyle\int_0^t \Big[-(\gamma_{12}+i(\omega_{12}-\omega))(\bar{L}_1-\bar{L}_2) \\
&+ \displaystyle\frac{ip^2}{\hbar}\Big((\bar{A}_1-\bar{A}_2)N_0+(\bar{A}_1-\bar{A}_2)\bar{M}_1+(\bar{M}_1-\bar{M}_2)\bar{A}_2\Big)\Big] \\
& \qquad (\xi-c(t-\theta),\theta)d\theta, \\
(\Phi_{\bar{M}_1} - \Phi_{\bar{M}_2})(\xi;t) &= \displaystyle\int_0^t \Big[-\gamma_{11}(\bar{M}_1-\bar{M}_2) \\
&+ \displaystyle\frac{2i}{\hbar}\Big((\bar{A}_1^*-\bar{A}_2^*)\bar{L}_1-(\bar{A}_1-\bar{A}_2)\bar{L}_1^* \\
& \quad +(\bar{L}_1-\bar{L}_2)\bar{A}_2^*-(\bar{L}_1-\bar{L}_2)^*\bar{A}_2\Big)\Big](\xi-c(t-\theta),\theta)d\theta.
\end{cases}
$$

Carrying on the same type of estimates than before, we obtain (after a possible restriction of the time interval) the fact that Φ is a contraction in the ball, we consider. This yields existence and uniqueness in X^3, i.e. the first part of the theorem. We show the continuity with respect to the initial data as in the case of Maxwell-Debye equations. ■

As for Maxwell-Debye equations, the existence time is the same in every H^s. This result is given by the theorem

Theorem 8 *Let* $(\varphi,\lambda,\mu)\in L^\infty(\xi;H^{1+\epsilon})\times L^\infty(\xi;H^{1+\epsilon})\times L^\infty(\xi;H^{1+\epsilon})$, $\epsilon>0$, *and* $(\bar{A},\bar{L},\bar{M})$ *be the maximal solution to Maxwell-Bloch equations in* $H^{1+\epsilon}$. *Let* $T_{1+\epsilon}$ *be its existence time. Let us moreover suppose that* $(\varphi,\lambda,\mu)\in L^\infty(\xi;H^s)\times L^\infty(\xi;H^s)\times L^\infty(\xi;H^s)$ *with* $s>1+\epsilon$, *then* $(\bar{A},\bar{L},\bar{M})$ *is solution to Maxwell-Bloch equation in* $(L^\infty(0,T_{1+\epsilon};H^s))^3$.

Proof :
The method is the same than in the case of Maxwell-Debye equations, but this time we are not dealing with a single equation.
We keep the initial form for the equation for A and we consider the equations for L and M after a method of the caracteristics, we choose a single caracteristic curve and we define $\tilde{A}$, $\tilde{L}$ and $\tilde{M}$ as for Maxwell-Debye equations,

then we obtain :

$$\left\{\begin{array}{lcl}\dfrac{\partial\tilde{A}}{\partial t}(t) &=& i\dfrac{c^2}{2\omega}\nabla_1^2\tilde{A}(t)-\kappa\tilde{A}(t)+\dfrac{i\omega}{2\epsilon_0}\tilde{L}(t),\\ \tilde{L}(t) &=& \lambda+\displaystyle\int_0^t\left[-(\gamma_{12}+i(\omega_{12}-\omega))\tilde{L}+\dfrac{ip^2}{c\hbar}(\tilde{A}N_0+\tilde{A}\tilde{M})\right](\theta)d\theta,\\ \tilde{M}(t) &=& \mu+\displaystyle\int_0^t\left[-\gamma_{11}\tilde{M}+\dfrac{2i}{\hbar}(\tilde{A}^*\tilde{L}-\tilde{A}\tilde{L}^*)\right](\theta)d\theta.\end{array}\right. \tag{3.8}$$

We multiply the first equation by J^sA and take the real part of the scalar product in L^2. This yields the first estimate

$$\frac{1}{2}\frac{d}{dt}\|\tilde{A}(t)\|_{H^s}^2+\kappa\|\tilde{A}(t)\|_{H^s}^2\leq C\|\tilde{L}(t)\|_{H^s}\|\tilde{A}(t)\|_{H^s},$$

and taking into account the fact that κ is positive,

$$\|\tilde{A}(t)\|_{H^s}\leq\|\varphi\|_{H^s}+C\int_0^t\|\tilde{L}(\theta)\|_{H^s}d\theta.$$

Then we compute the H^s norm of the two equations for L and M, together with the fact that the $H^{1+\epsilon}$ norm is bounded on the interval of time, we consider, which yields :

$$\begin{array}{rcl}\|\tilde{L}(t)\|_{H^s} &\leq& \|\lambda\|_{H^s}\\ &+& C\displaystyle\int_0^t\{\|\tilde{L}(\theta)\|_{H^s}+\|\tilde{A}(\theta)\|_{H^s}+\|\tilde{M}(\theta)\|_{H^s}\}d\theta,\\ \|\tilde{M}(t)\|_{H^s} &\leq& \|\mu\|_{H^s}\\ &+& C\displaystyle\int_0^t\{\|\tilde{M}(\theta)\|_{H^s}+\|\tilde{A}(\theta)\|_{H^s}+\|\tilde{L}(\theta)\|_{H^s}\}d\theta.\end{array}$$

Then we get

$$\begin{array}{rcl}\|\tilde{A}(t)\|_{H^s}+\|\tilde{L}(t)\|_{H^s}+\|\tilde{M}(t)\|_{H^s} &\leq& (\|\varphi\|_{H^s}+\|\lambda\|_{H^s}+\|\mu\|_{H^s})\\ &+& C\displaystyle\int_0^t\left(\|\tilde{A}(\theta)\|_{H^s}+\|\tilde{L}(\theta)\|_{H^s}+\|\tilde{M}(\theta)\|_{H^s}\right)d\theta\end{array}$$

and by Gronwall's lemma, the existence time of the solutions to Maxwell-Bloch equations for initial data in H^s is the same as in $H^{1+\epsilon}$. ■

4 Conclusion

Except for the adiabatic Maxwell-Bloch equations, all the results we obtain are local in time. A natural question would be the global well-posedness of the Cauchy problem both for Maxwell-Debye and Maxwell-Bloch systems. This problem is usually studied for a H^1 regularity. The case of Maxwell-Bloch equations seems very difficult since there is no conservation law and we do not dispose of any H^1 local well-posedness result. Concerning Maxwell-Debye equations the situation is a little better since the L^2 norm of A is conserved and we have local H^1 results. Unfortunately we do not have any H^1 conservation law. All attempts to overcome this difficulty have been vain until now mainly because of the non local in time character of the equations and the fact that many terms we want to estimate have no definite sign. This question is also connected to that of finite time blow-up.
Other results are in preparation as the existence of solitary waves.

The models we present here are among the simplest in nonlinear optics. There exists numerous other systems which are similar to Maxwell-Bloch one but with a larger number of possible excited states. For a three-level atom, we obtain coupled Schrödinger equations with six transport equations :

$$
\begin{aligned}
&\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)A_1+\frac{i}{2k}\nabla_1^2 A_1+\alpha_1 A_1=i\frac{\omega_1}{2\epsilon_0 c}n_a p_{1m}\sigma_{m1},\\
&\left(\pm\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)A_2+\frac{i}{2k}\nabla_1^2 A_2+\alpha_2 A_2=i\frac{\omega_2}{2\epsilon_0 c}n_a p_{2m}\sigma_{m2},\\
&\frac{\partial}{\partial t}\sigma_{11}+\gamma_{\parallel}^1(\sigma_{11}-\sigma_{11}^0)=\frac{i}{\hbar}A_1^* p_{1m}\sigma_{m1}-\frac{i}{\hbar}A_1 p_{m1}\sigma_{1m},\\
&\frac{\partial}{\partial t}\sigma_{22}+\gamma_{\parallel}^2(\sigma_{22}-\sigma_{22}^0)=\frac{i}{\hbar}A_2^* p_{2m}\sigma_{m2}-\frac{i}{\hbar}A_2 p_{m2}\sigma_{2m},\\
&\frac{\partial}{\partial t}\sigma_{mm}+\gamma_{\parallel}^3(\sigma_{mm}-\sigma_{mm}^0)=\frac{i}{\hbar}(A_1 p_{1m}\sigma_{1m}+A_2 p_{m2}\sigma_{m2})\\
&\qquad\qquad\qquad\qquad\qquad\quad -\frac{i}{\hbar}(A_1^* p_{1m}\sigma_{1m}+A_2^* p_{2m}\sigma_{2m}),\\
&\frac{\partial}{\partial t}\sigma_{1m}+(\gamma_{1m}+i(\omega_1+\omega_{1m}))\sigma_{1m}=\frac{i}{\hbar}p_{1m}A_1^*(\sigma_{mm}-\sigma_{11})-\frac{i}{\hbar}p_{2m}A_2^*\sigma_{12},\\
&\frac{\partial}{\partial t}\sigma_{2m}+(\gamma_{2m}+i(\omega_2+\omega_{2m}))\sigma_{2m}=\frac{i}{\hbar}p_{2m}A_2^*(\sigma_{mm}-\sigma_{22})-\frac{i}{\hbar}p_{1m}A_1^*\sigma_{21},\\
&\frac{\partial}{\partial t}\sigma_{12}+(\gamma_{12}+i(\omega_1-\omega_2+\omega_{12}))\sigma_{12}=\frac{i}{\hbar}p_{1m}A_1^*\sigma_{m2}-\frac{i}{\hbar}p_{m2}A_2\sigma_{1m}.
\end{aligned}
$$

Maxwell-Debye equations also admit generalizations. This is the case when we replace the monodirectional wave by two counter-propagating waves : $\vec{E} = \hat{e}\left(A_1 e^{i(kz-\omega t)} + A_2 e^{-i(kz-\omega t)} + c.c.\right)$. Then we obtain two Schrödinger equations coupled with two delay equations :

$$\begin{aligned}
\left(\frac{\partial}{\partial z} + \frac{n_0}{c}\frac{\partial}{\partial t}\right) A_1 - \frac{i}{2k_0}\nabla_1^2 A_1 &= -i\frac{\omega_0}{c}\left(\delta n_0 A_1 + \delta n_1 A_2\right), \\
\left(-\frac{\partial}{\partial z} + \frac{n_0}{c}\frac{\partial}{\partial t}\right) A_2 - \frac{i}{2k_0}\nabla_1^2 A_2 &= +i\frac{\omega_0}{c}\left(\delta n_0 A_2 + \delta n_1^* A_1\right), \\
\tau\frac{\partial}{\partial t}\delta n_0 + \delta n_0 &= n_2\left(A_1 A_1^* + A_2 A_2^*\right), \\
\tau\frac{\partial}{\partial t}\delta n_1 + \delta n_1 &= n_2 A_1 A_2^*.
\end{aligned}$$

For these equations, the study of the Cauchy problem may be done following the model of those carried out in the present article.

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