FROM NEWTON'S CRADLE TO THE DISCRETE P-SCHRÖDINGER EQUATION[*]

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Abstract. We investigate the dynamics of a chain of oscillators coupled by fully-nonlinear interaction potentials. This class of models includes Newton's cradle with Hertzian contact interactions between neighbors. By means of multiple-scale analysis, we give a rigorous asymptotic description of small amplitude solutions over large times. The envelope equation leading to approximate solutions is a discrete p-Schrödinger equation. Our results include the existence of long-lived breather solutions to the original model. For a large class of localized initial conditions, we also estimate the maximal decay of small amplitude solutions over long times.

Key words. Newton's cradle, nonlinear lattice, Hertzian interactions, multiple-scale analysis, discrete p-Schrödinger equation, breather

AMS subject classifications. 37K60, 70F45, 70K70, 74J30

1. Introduction. Newton's cradle is a nonlinear mechanical system consisting of a chain of identical beads suspended from a bar by inelastic strings (see figure 1.1). All beads behave like pendula in the absence of contact with nearest neighbors, i.e. they perform time-periodic oscillations in a local confining potential $\Phi(x) = \frac{1}{2}x^2 + O(x^4)$ due to gravity. More generally, the local potential Φ may account for different types of stiff attachments [JKC13] or an elastic matrix surrounding the beads [Lav12, HCRVMK13]. Mechanical constraints between touching beads can be described by the Hertzian interaction potential $V(r) = \frac{k}{1+\alpha}(-r)_+^{\alpha+1}$, where $(a)_+ = \max(a, 0)$, k depends on the ball radius and material and $\alpha = 3/2$. The dynamical equations read in dimensionless form [HDWM04]

$$(1.1) \qquad \ddot{x}_n + \Phi'(x_n) = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

x_n being the horizontal displacement of the nth bead from its equilibrium position at which the pendulum is vertical.
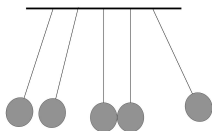

FIG. 1.1. *Schematic representation of Newton's cradle.*

Contact interactions between beads induce a nonlinear coupling, which can lead to complex dynamical phenomena like the propagation of solitary waves [Nes01, FW94,

Mac99, EP05, SK12, JKC13], modulational instabilities [Jam11, JKC13, BTJKPD10] and the excitation of spatially localized stationary (time-periodic) or moving breathers [TBKJPD10, Jam11, SHVM12, JCK12, JKC13]. For small amplitude oscillations, it has been recently argued that such dynamical phenomena can be captured by the discrete p-Schrödinger (DpS) equation

$$(1.2) \qquad 2i\tau_0 \dot{A}_n = (A_{n+1} - A_n)\,|A_{n+1} - A_n|^{\alpha-1} - (A_n - A_{n-1})\,|A_n - A_{n-1}|^{\alpha-1},$$

with some time constant τ_0 depending on α. More precisely, static breather solutions to (1.1) were numerically computed in [JKC13] and compared to approximate solutions of the form

$$(1.3) \qquad\qquad\qquad x_n^{\mathrm{app}}(t) = 2\,\varepsilon\,\mathrm{Re}\,[\,A_n(\varepsilon^{\alpha-1}t)\,e^{it}\,],$$

where $\varepsilon \ll 1$ and A_n denotes a breather solution to the DpS equation (1.2), which depends on the slow time variable $\tau = \varepsilon^{\alpha-1}t$. For small amplitudes, the Ansatz (1.3) was found to approximate breather solutions to (1.1) with good accuracy [JKC13], and the same property was established in [Jam11] for periodic traveling waves. Moreover, a small amplitude velocity perturbation at the boundary of a semi-infinite chain (1.1) generates a traveling breather whose profile is qualitatively close to (1.3), where A_n corresponds to a traveling breather solution of the DpS equation [SHVM12, JKC13].

In this paper, we put the relation between the original lattice (1.1) and the DpS equation onto a rigorous footing. Our main result can be stated as follows (a more precise statement extended to more general potentials is given in theorem 2.15, section 2.6). Given a smooth (C^2) solution $A = (A_n(\tau))_{n\in\mathbb{Z}}$ to (1.2) defined for $\tau \in [0, T]$, if an initial condition for (1.1) is $O(\varepsilon^\alpha)$-close to the Ansatz (1.3) at $t = 0$, then the corresponding solution to (1.1) remains $O(\varepsilon^\alpha)$-close to the approximation (1.3) on $O(\varepsilon^{1-\alpha})$ time scales. These error estimates hold in the usual sequence spaces ℓ_p with $1 \leq p \leq +\infty$. In addition, if A is a global and bounded solution to (1.2) in $\ell_p(\mathbb{Z})$ and $\delta \in (1, \alpha)$ is fixed, then the same procedure yields $O(|\ln\varepsilon|\,\varepsilon^\delta)$-close approximate solutions up to times $t = O(|\ln\varepsilon|\,\varepsilon^{1-\alpha})$ (theorem 2.20). Moreover, similar estimates allow one to approximate the evolution of *all* sufficiently small initial data to (1.1) in $(\ell_p(\mathbb{Z}))^2$ (theorems 2.19 and 2.21).

Two applications of the above error estimates are presented. Firstly, for all non-trivial solutions A of the DpS equation in $\ell_2(\mathbb{Z})$ we demonstrate that $\inf_{\tau\in\mathbb{R}}\|A(\tau)\|_\infty > 0$, i.e. solutions associated with localized (square-summable) initial conditions do not completely disperse. Using this result and the above error bounds, we estimate the maximal decay of small amplitude solutions to (1.1) over long times for a large class of localized initial conditions. Secondly, from a breather existence theorem proved in [JS13] for the DpS equation, we deduce the existence of stable small amplitude "long-lived" breather solutions to equation (1.1), which remain close to time-periodic and spatially localized oscillations over long times. This result completes a previous existence theorem for stationary breather solutions of (1.1) proved in [JCK12], which was restricted to anharmonic on-site potentials Φ and small values of the coupling constant k. More generally, the present justification of the DpS equation is also useful in the context of numerical simulations of granular chains. Indeed, the DpS system is much easier to simulate than equation (1.1) due to the fact that fast local oscillations have been averaged, which allows to perform larger numerical integration steps.

Our results are in the same spirit as rigorous derivations of the continuum cubic nonlinear Schrödinger or Davey-Stewartson equations, which approximate the evolution of the envelope of slowly modulated normal modes in a large class of nonlinear

lattices [GM04, GM06, BCP09, Sch10] and hyperbolic systems [DJMR95, JMR98, Sch98, Col02, CL04]. In addition, our extension of the error bounds up to times τ growing logarithmically in ε (theorem 2.20) is reminiscent of refined approximations of nonlinear geometric optics derived in [LR00]. A specificity of our result is the spatially discrete character of the amplitude equation (1.2), which allows one to describe nonlinear waves with rather general spatial behaviors (see theorem 2.19 and 2.21). Another particular feature of our study is the fact that potentials and nonlinearities can have limited smoothness, so that high-order corrections seem hardly available.

The outline of the paper is as follows. In section 2.1, we introduce a generalized version of system (1.1) involving more general potentials, which is reformulated as a first order differential equation in $(\ell_p(\mathbb{Z}))^2$. Section 2.2 presents elementary properties of periodic solutions to the linearized evolution problem, which will be used in the subsequent analysis. The well-posedness of the Cauchy problem for the nonlinear evolution equation is established in section 2.3. In section 2.4, we perform a formal multiple-scale analysis, yielding approximate solutions to (1.1) consisting of slow modulations of periodic solutions. In this approximation, the leading term (1.3) is supplemented by a higher-order corrector. Some qualitative properties of the amplitude equation are detailed in section 2.5, including well-posedness and the study of spatially localized solutions. The main results on the justification of the multiple-scale analysis from error bounds are stated in section 2.6. The error bounds are derived in sections 2.6 and 3 (the later contains the proof of theorems 2.15 and 2.20, which are mainly based on Gronwall estimates). Section 4 provides a discussion of the above results and points out some open problems, and some technical results are detailed in the appendix.

2. Dynamical equations and multiple-scale analysis.

2.1. Nonlinear lattice model. We first introduce a more general version of system (1.1) incorporating a larger class of potentials. We consider an interaction potential $V \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ of the form $V = V_\alpha + W$, where

$$(2.1) \qquad V_\alpha(r) = \begin{cases} \frac{k_-}{1+\alpha} |r|^{\alpha+1} & \text{for } r \leq 0, \\ \frac{k_+}{1+\alpha} r^{\alpha+1} & \text{for } r \geq 0, \end{cases}$$

and $\alpha > 1$, $(k_-, k_+) \neq (0, 0)$. In addition, W is a higher order correction satisfying

$$(2.2) \qquad W'(0) = 0, \quad W''(r) = \mathcal{O}(|r|^{\alpha-1+\beta}) \quad \text{as } r \to 0,$$

for some constant $\beta > 0$. We can therefore write $W'(r) = |r|^\alpha \rho(r)$ where $\rho(r) = \mathcal{O}(|r|^\beta)$ as $r \to 0$. Under the above assumptions, the principal part of V satisfies $V'_\alpha(\lambda r) = \lambda^\alpha V'_\alpha(r)$ for all $r \in \mathbb{R}$ and $\lambda > 0$. Note that one recovers the classical Hertzian potential by fixing $k_- = k > 0$, $k_+ = 0$ and $W = 0$. The case $k_- = k_+$ and $W = 0$ corresponds to an homogeneous even interaction potential.

In addition, the local potential Φ is assumed of the form $\Phi(x) = \frac{1}{2}x^2 + \phi(x)$, where $\phi \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ satisfies

$$(2.3) \qquad \phi'(0) = 0, \quad \phi''(x) = \mathcal{O}(|x|^{\alpha-1+\gamma}) \quad \text{as } x \to 0,$$

for some constant $\gamma > 0$. We have therefore $\phi'(x) = |x|^\alpha \chi(x)$ with $\chi(x) = \mathcal{O}(|x|^\gamma)$ as $x \to 0$. The particular case of an harmonic on-site potential Φ is obtained by fixing $\phi = 0$.

The dynamical equations read

$$(2.4) \qquad \ddot{x}_n + x_n = F(x)_n, \quad n \in \mathbb{Z},$$

where

$$F(x)_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}) - \phi'(x_n), \quad n \in \mathbb{Z}.$$

From the above assumptions, the leading order nonlinear terms of (2.4) originate from the interaction potential V_α. We denote $x = (x_n)_{n \in \mathbb{Z}}$ and address solutions $x(t) \in \ell_p \equiv \ell_p(\mathbb{Z}, \mathbb{R})$. Throughout the paper we assume $p \in [1, \infty]$ unless explicitly stated.

In what follows we reformulate equation (2.4) as a first order differential equation governing the $X = (x, \dot{x})^T$ variable where $X(t) \in \ell_p^2$. This equation reads

$$(2.5) \qquad \dot{X} = J\,X + G(X),$$

where

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\,\mathrm{Id} & 0 \end{pmatrix}, \quad G(X_1, X_2) = \begin{pmatrix} 0 \\ F(X_1) \end{pmatrix},$$

and Id is the identity map in ℓ_p. Using the usual difference operators $(\delta^+ x)_n = x_{n+1} - x_n$ and $(\delta^- x)_n = x_n - x_{n-1}$, we can write F in a compact form:

$$F(x) = \delta^+ V'(\delta^- x) - \phi'(x).$$

The Banach spaces ℓ_p^2 are equipped with the following norms

$$(2.6) \quad \|X\|_p = \left(\sum_{n \in \mathbb{Z}} (x_n^2 + \dot{x}_n^2)^{p/2} \right)^{1/p} \text{ if } 1 \le p < \infty, \quad \|X\|_\infty = \sup_{n \in \mathbb{Z}} (x_n^2 + \dot{x}_n^2)^{1/2}.$$

The map G is smooth and fully-nonlinear in ℓ_p^2, as shown by the following lemma. Below and in the rest of the paper, we use the abbreviations *c.n.d.f.* for continuous non-decreasing functions and *c.n.i.f.* for continuous non-increasing functions.

Lemma 2.1. *The map $G \in \mathcal{C}^1(\ell_p^2, \ell_p^2)$ satisfies*

$$(2.7) \qquad \|G(X)\|_p = \mathcal{O}(\|X\|_p^\alpha), \quad \|DG(X)\|_{\mathcal{L}(\ell_p^2)} = \mathcal{O}(\|X\|_p^{\alpha-1})$$

when $X \to 0$ in ℓ_p^2. Moreover, G and DG are bounded on bounded sets in ℓ_p^2, and there exist c.n.d.f. C_G, C_D such that for all $X \in \ell_p^2$

$$(2.8) \qquad \|G(X)\|_p \le C_G(\|X\|_\infty)\, \|X\|_p, \quad \|DG(X)\|_{\mathcal{L}(\ell_p^2)} \le C_D(\|X\|_\infty).$$

Proof. It is a classical result that the functions $f = V'$ or $f = \phi'$ can be viewed as smooth operators on $\ell_p(\mathbb{Z}, \mathbb{R})$ via $(f(x))_n \overset{\text{def}}{=} f(x_n)$. The property $V' \in \mathcal{C}^1(\ell_p, \ell_p)$ follows from the continuous embedding $\ell_p \subset \ell_\infty$, the fact that $V' \in \mathcal{C}^1(\mathbb{R})$, $V'(0) = 0$ and V'' is uniformly continuous on compact intervals. The property $\phi' \in \mathcal{C}^1(\ell_p, \ell_p)$ follows from the same arguments. These properties imply that $F \in \mathcal{C}^1(\ell_p, \ell_p)$ and $G \in \mathcal{C}^1(\ell_p^2, \ell_p^2)$.

Moreover, standard estimates yield the following inequalities for $f = V', \phi'$ and all $x \in \ell_p$,

$$(2.9) \qquad \|Df(x)\|_{\mathcal{L}(\ell_p)} \leq M_f(\|x\|_\infty), \quad \|f(x)\|_p \leq M_f(\|x\|_\infty) \, \|x\|_p,$$

where $M_f(d) = \sup_{[0,d]} |f'|$ defines a c.n.d.f. of $d \in \mathbb{R}^+$. These estimates yield the bounds (2.8), hence G and DG are bounded on bounded sets in ℓ_p^2 due to the continuous embedding $\ell_p \subset \ell_\infty$. Moreover, (2.7) follows by combining (2.9) with the bounds

$$M_{V'}(r) = \mathcal{O}(r^{\alpha-1}), \quad M_{\phi'}(r) = \mathcal{O}(r^{\alpha-1+\gamma}), \quad r \to 0^+,$$

which originate from properties (2.1), (2.2) and (2.3). $\qquad\qquad\qquad\square$

In section 2.3, we prove the local well-posedness in ℓ_p^2 of the Cauchy problem associated with (2.5), and its global well-posedness for $1 \leq p \leq 2$ when $V \geq 0$ and $\phi \geq 0$. To obtain global solutions, we use the fact that the Hamiltonian

$$(2.10) \qquad H = \sum_{n\in\mathbb{Z}} \frac{1}{2}\dot{x}_n^2 + \frac{1}{2}x_n^2 + \phi(x_n) + V(x_{n+1} - x_n)$$

is a conserved quantity of (2.4).

2.2. Periodic solutions to the linearized equation. In this section, we consider the time-periodic solutions to the linearized dynamical equations which constitute the basic pattern slowly modulated in section 2.4. We present some elementary properties of these solutions, solve the nonhomogeneous linearized equations and compute the associated solvability conditions. This result will be used in section 2.4 to derive the amplitude equation (1.2) as a solvability condition and obtain the expression of a higher-order corrector to approximation (1.3), following a usual multiple-scale perturbation scheme (see e.g. [SS99], section 1.1.3).

Equation (2.4) linearized at $x_n = 0$ reads

$$(2.11) \qquad \ddot{x}_n + x_n = 0, \quad n \in \mathbb{Z},$$

or equivalently

$$(2.12) \qquad X' = J\,X.$$

Its solutions are 2π-periodic and take the form

$$\begin{pmatrix} x_n(t) \\ \dot{x}_n(t) \end{pmatrix} = a_n e^{it} e_1 + \text{c.c.},$$

where c.c. denotes the complex conjugate, $e_{\pm 1} = (1/\sqrt{2})(1, \pm i)^T$ and $a_n = (x_n(0) - i\dot{x}_n(0))/\sqrt{2}$. Moreover, assuming $X(0) \in \ell_p^2$ corresponds to imposing $(a_n)_{n\in\mathbb{Z}} \in \ell_p(\mathbb{Z}, \mathbb{C})$. In a more compact form, the solution $X = (x, \dot{x})^T$ to equation (2.12) with initial condition $X(0) = X^0 = (x^0, \dot{x}^0)^T$ reads

$$(2.13) \qquad X(t) = e^{Jt} X^0 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} X^0$$

$$(2.14) \qquad\qquad = (\pi_1 X^0)\, e^{it} e_1 + (\pi_{-1} X^0)\, e^{-it} e_{-1}$$

where $\pi_{\pm 1} \in \mathcal{L}(\ell_p^2, \ell_p(\mathbb{Z}, \mathbb{C}))$ are defined by

$$\pi_{\pm 1} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \frac{1}{\sqrt{2}} (x \mp i\dot{x}),$$

so that $\pi_1 X^0 = (a_n)_{n \in \mathbb{Z}}$.

Remark 2.2. One can notice that $x_n^2 + \dot{x}_n^2$ is a conserved quantity of (2.11) for all $n \in \mathbb{Z}$. With the choice of norm (2.6), $\|X(t)\|_p$ is conserved along evolution for solutions to (2.12) and it follows that $\|e^{Jt}\|_{\mathcal{L}(\ell_p^2)} = 1$ for all $t \in \mathbb{R}$. Moreover we have $\|X(t)\|_p = \|X^0\|_p = \sqrt{2}\, \|\pi_{\pm 1} X^0\|_p$.

In what follows we consider the nonhomogeneous linearized equation

$$(2.15) \qquad\qquad \dot{X} = J\,X + U,$$

where $U : \mathbb{R} \to \ell_p^2$ is 2π-periodic, and derive compatibility conditions on U allowing for the existence of 2π-periodic solutions to (2.15). We denote $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ the periodic interval $[0, 2\pi]$ and consider the function spaces $\mathbb{X}^0 = \mathcal{C}^0(S^1, \ell_p^2)$ and $\mathbb{X}^1 = \mathcal{C}^1(S^1, \ell_p^2)$ endowed with their usual uniform topology.

Lemma 2.3. *Let $U \in \mathbb{X}^0$. The differential equation (2.15) has a solution $X \in \mathbb{X}^1$ if and only if*

$$(2.16) \qquad\qquad \pi_1 \left(\int_0^{2\pi} e^{-it} U(t)\, \mathrm{d}t \right) = 0.$$

Proof. Given $U \in \mathcal{C}^0(\mathbb{R}, \ell_p^2)$, the differential equation (2.15) with initial condition $X(0) = X^0 \in \ell_p^2$ has a unique solution $X \in \mathcal{C}^1(\mathbb{R}, \ell_p^2)$ given by the Duhamel integral

$$(2.17) \qquad\qquad X(t) = e^{Jt} X^0 + \int_0^t e^{J(t-s)} U(s)\, \mathrm{d}s.$$

Now let us assume $U \in \mathbb{X}^0$. In this case, X is 2π-periodic iff $X(2\pi) = X(0)$. Since $e^{J 2\pi} = \mathrm{Id}$, this condition is realized when

$$\int_0^{2\pi} e^{-Js} U(s)\, \mathrm{d}s = 0,$$

which is equivalent to condition

$$(2.18) \qquad\qquad \pi_{\pm 1} \left(\int_0^{2\pi} e^{\mp it} U(t)\, \mathrm{d}t \right) = 0$$

due to identity (2.14). Since U is real and $\overline{\pi_1 z} = \pi_{-1} \bar{z}$, condition (2.18) reduces simply to (2.16). $\qquad\square$

The above results yield the following splitting of $\mathbb{X}^0$.

Lemma 2.4. *The operator $\partial_t - J$ maps $\mathbb{X}^1$ to $\mathbb{X}^0$, and we have the splitting $\mathbb{X}^0 = \ker(\partial_t - J) \oplus \mathrm{range}(\partial_t - J)$. The corresponding projector P on $\ker(\partial_t - J)$ along $\mathrm{range}(\partial_t - J)$ reads*

$$PX = \zeta(X) e^{it} e_1 + \bar{\zeta}(X) e^{-it} e_{-1},$$

where $\zeta \in \mathcal{L}(\mathbb{X}^0, \ell_p(\mathbb{Z}, \mathbb{C}))$ is defined by

$$\zeta(X) = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} \pi_1 X(t)\, \mathrm{d}t.$$

Proof. It is clear that $P \in \mathcal{L}(\mathbb{X}^0)$ defines a projection and range $P = \ker(\partial_t - J)$ by identity (2.14). Moreover, condition (2.16) shows that $\ker P = \text{range}(\partial_t - J)$, hence $\mathbb{X}^0 = \text{range} \, P \oplus \ker P = \ker(\partial_t - J) \oplus \text{range}(\partial_t - J)$. $\qquad\square$

Now one can deduce the following result from expression (2.17) and lemma 2.4.

Lemma 2.5. *For all $U \in \mathbb{X}^0$ satisfying (2.18) (or equivalently $PU = 0$), equation (2.15) has a unique solution in $\mathbb{X}^1 \cap \text{range}(\partial_t - J)$ given by*

$$(2.19) \qquad X(t) = (KU)(t) = (I - P) \int_0^t e^{J(t-s)} U(s) \, \mathrm{d}s.$$

Moreover, the linear operator $\mathcal{K} \overset{def}{=} K(I - P) : \mathbb{X}^0 \to \mathbb{X}^1$ is bounded.

2.3. Well-posedness of the nonlinear evolution problem. The following result ensures the local well-posedness of the Cauchy problem for (2.5) in ℓ_p^2, and its global well-posedness for positive potentials when $1 \le p \le 2$. In addition we derive a crude lower bound on the maximal existence times for small initial data (estimate (2.20)).

Lemma 2.6. *For all initial condition $X(0) = X^0 \in \ell_p^2$, equation (2.5) admits a unique solution $X \in \mathcal{C}^2((t_-, t_+), \ell_p^2)$, defined on a maximal interval of existence (t_-, t_+) depending a priori on X^0 (with $t_- < 0 < t_+$). In addition, there exists $T_0 > 0$ such that for $\|X(0)\|_p$ small enough*

$$(2.20) \qquad t_+ > T_0 \|X(0)\|_p^{1-\alpha}, \quad t_- < -T_0 \|X(0)\|_p^{1-\alpha}.$$

Moreover, for $p \in [1, 2]$, if $V \ge 0$ and $\phi \ge 0$ then $(t_-, t_+) = \mathbb{R}$ and $X \in L^\infty(\mathbb{R}, \ell_2^2)$.

Proof. Since G is $\mathcal{C}^1$ in ℓ_p^2, it follows that the Cauchy problem for (2.5) is locally well-posed in ℓ_p^2 (see e.g. [Zei95], section 4.9). More precisely, for all initial condition $X(0) = X^0 \in \ell_p^2$, equation (2.5) admits a unique solution $X \in \mathcal{C}^1((t_-, t_+), \ell_p^2)$, defined on a maximal interval of existence (t_-, t_+) depending *a priori* on X^0 (with $t_- < 0 < t_+$). Then a bootstrap argument yields $X \in \mathcal{C}^2((t_-, t_+), \ell_p^2)$.

Now let us prove (2.20) by Gronwall-type estimates. Equation (2.5) can be expressed in Duhamel form

$$(2.21) \qquad X(t) = e^{Jt} X(0) + \int_0^t e^{J(t-s)} G(X(s)) \, \mathrm{d}s,$$

where we recall that $\|e^{Jt}\|_{\mathcal{L}(\ell_p^2)} = 1$. In addition, by lemma 2.1 there exist $R, \lambda > 0$ such that

$$(2.22) \qquad \|G(X)\|_p \le \lambda \|X\|_p^\alpha$$

for all $X \in \ell_p^2$ with $\|X\|_p \le R$. Denote by ε a small parameter and fix $\|X(0)\|_p = \varepsilon$. From the above properties we deduce

$$\|X(t)\|_p \le \varepsilon + \lambda \int_0^t \|X(s)\|_p^\alpha \, \mathrm{d}s, \quad 0 \le t \le t_1,$$

where $t_1 = \sup\{ t \ge 0, \|X(t)\|_p \le R \}$. This yields the estimate

$$(2.23) \qquad \|X(t)\|_p \le \varrho_{\varepsilon,\lambda}(t), \quad 0 \le t \le t_1,$$

where $\varrho_{\varepsilon,\lambda}$ is the solution to the differential equation

$$(2.24) \qquad \qquad \varrho' = \lambda\,\varrho^{\alpha}, \quad \varrho(0) = \varepsilon$$

whose explicit form is

$$(2.25) \qquad \qquad \varrho_{\varepsilon,\lambda}(t) = \varepsilon\,[\,1 - (\alpha - 1)\,\lambda\,\varepsilon^{\alpha-1}t\,]^{\frac{1}{1-\alpha}}.$$

We note that $\varrho_{\varepsilon,\lambda}$ blows up at $t = \varepsilon^{1-\alpha}\,[\lambda\,(\alpha - 1)]^{-1}$. Consequently, we fix $\theta \in (0,1)$ and introduce $t_{\varepsilon} = T_0\,\varepsilon^{1-\alpha}$ with $T_0 = (1-\theta)\,[\lambda\,(\alpha - 1)]^{-1}$, and hence for all $t \in [0, t_{\varepsilon}]$ we have

$$(2.26) \qquad \qquad \varrho_{\varepsilon,\lambda}(t) \le \varrho_{\varepsilon,\lambda}(t_{\varepsilon}) = \varepsilon\,\theta^{\frac{1}{1-\alpha}}.$$

Now let us assume $\|X(0)\|_p = \varepsilon \le \theta^{\frac{1}{\alpha-1}}R$. By estimates (2.23) and (2.26) we have then

$$(2.27) \qquad \qquad \|X(t)\|_p \le \theta^{\frac{1}{1-\alpha}}\varepsilon \quad \text{for } |t| \le t_{\varepsilon},$$

where the estimate for $t \le 0$ is the same as for $t \ge 0$ due to the time-reversibility of (2.5) inherited from (2.4). Consequently, $X(t)$ is defined and $\mathcal{O}(\varepsilon)$ in ℓ_p^2 at least for $t \in [-t_{\varepsilon}, t_{\varepsilon}]$, which proves (2.20).

In addition, the existence of a global solution to (2.5) in ℓ_2^2 can be proved when $V \ge 0$ and $\phi \ge 0$, using the fact that the Hamiltonian (2.10) is a conserved quantity of (2.4). Indeed, for all initial condition $X(0) = X^0 \in \ell_2^2$ and for all $t \in (t_-, t_+)$ we have in that case

$$(2.28) \qquad \|X(t)\|_2^2 \le 2H = \|X^0\|_2^2 + 2\sum_{n\in\mathbb{Z}}\big(\phi(x_n(0)) + V(x_{n+1}(0) - x_n(0))\big),$$

and thus $X \in L^{\infty}(\mathbb{R}, \ell_2^2)$. Lemma 2.1 ensures that G and DG are bounded on bounded sets in ℓ_2^2. Together with the uniform bound (2.28) on the solution X, this property implies that $(t_-, t_+) = \mathbb{R}$ (see e.g. [RS75], theorem X.74).

Similar arguments can be used for global well-posedness in ℓ_p^2 with $p \in [1,2)$, except one uses the fact that $\|X(t)\|_p$ is bounded on bounded time intervals. Indeed, the following estimate follows from equation (2.21) and lemma 2.1

$$\|X(t)\|_p \le \|X(0)\|_p + \int_0^t C_G(\|X(s)\|_\infty)\,\|X(s)\|_p\,\mathrm{d}s,$$

where C_G is a c.n.d.f. Then using the continuous embedding $\ell_2 \subset \ell_\infty$ and estimate (2.28), we find

$$\|X(t)\|_p \le \|X(0)\|_p + C_G(\sqrt{2H})\int_0^t \|X(s)\|_p\,\mathrm{d}s,$$

hence by Gronwall's lemma

$$\|X(t)\|_p \le e^{C_G(\sqrt{2H})\,t}\,\|X(0)\|_p.$$

This shows that $\|X(t)\|_p$ is bounded on bounded time intervals, which completes the proof. $\qquad\square$

2.4. Multiple-scale expansion. In this section, we perform a multiple-scale analysis in order to obtain approximate solutions to equation (2.5). These approximations consist of slow time modulations of small 2π-periodic solutions to the previously analyzed linearized equation (2.12).

To determine the relevant time scales, we denote by ε a small parameter and fix $\|X(0)\|_p = \varepsilon$. As previously seen in the proof of lemma 2.6, the solution $X(t)$ of (2.5) is defined and $\mathcal{O}(\varepsilon)$ in ℓ_p^2 at least on long time scales $t \in [-t_\varepsilon, t_\varepsilon]$ with $t_\varepsilon = T_0\,\varepsilon^{1-\alpha}$. Considering the Duhamel form (2.21) of (2.5) when $t \approx t_\varepsilon$, the integral term at the right side of (2.21) is $\mathcal{O}(t_\varepsilon\,\varepsilon^\alpha)$, so that both terms are $\mathcal{O}(\varepsilon)$ and contribute "equally" to $X(t)$.

We therefore consider the slow time $\tau = \varepsilon^{\alpha-1}t$ in addition to the fast time variable t, and look for slowly modulated periodic solutions involving these two time scales:

$$X(t) = \varepsilon Y(\tau, t)_{|\tau=\varepsilon^{\alpha-1}t},$$

Y being 2π-periodic in the fast variable t. Injecting this Ansatz in Equation (2.5), we obtain

$$(2.29) \qquad (\partial_t - J)Y - \frac{1}{\varepsilon}G(\varepsilon Y) + \varepsilon^{\alpha-1}\partial_\tau Y = 0.$$

Considering the class of potentials described in section 2.1, one can write

$$(2.30) \qquad \frac{1}{\varepsilon}W'(\varepsilon x) = \varepsilon^{\alpha-1}r_\varepsilon(x), \quad \frac{1}{\varepsilon}\phi'(\varepsilon x) = \varepsilon^{\alpha-1}\eta_\varepsilon(x),$$

with $r_\varepsilon(x) = |x|^\alpha\rho(\varepsilon x) = \mathcal{O}(\varepsilon^\beta\,|x|^{\alpha+\beta})$ and $\eta_\varepsilon(x) = |x|^\alpha\chi(\varepsilon x) = \mathcal{O}(\varepsilon^\gamma\,|x|^{\alpha+\gamma})$ when $\varepsilon \to 0$. Hence we can split the nonlinear terms of (2.29) in the following way,

$$(2.31) \qquad \frac{1}{\varepsilon}G(\varepsilon Y) \equiv \varepsilon^{\alpha-1}G_\alpha(Y) + \varepsilon^{\alpha-1}R_\varepsilon(Y),$$

where for all $Y = (Y_1, Y_2)^T$,

$$G_\alpha(Y) = \begin{pmatrix} 0 \\ \delta^+ V'_\alpha(\delta^- Y_1) \end{pmatrix}, \quad R_\varepsilon(Y) = \begin{pmatrix} 0 \\ \delta^+ r_\varepsilon(\delta^- Y_1) - \eta_\varepsilon(Y_1) \end{pmatrix}.$$

Moreover, for all $Y \in \ell_p^2$ we have $\lim_{\varepsilon\to 0} R_\varepsilon(Y) = 0$ thanks to the assumptions made on r_ε and η_ε (see definition (2.30) and properties (2.2) and (2.3)). More precisely, using the second estimate of (2.9), there exist $C > 0$ and a c.n.i.f. ε_0 such that for all $Y \in \ell_p^2$ and $\varepsilon < \varepsilon_0(\|Y\|_\infty)$,

$$(2.32) \qquad \|R_\varepsilon(Y)\|_p \le C\,\varepsilon^{\min(\beta,\gamma)}\,\|Y\|_p^\alpha\,(\|Y\|_p^\beta + \|Y\|_p^\gamma).$$

Now let us consider $\mathcal{Y}(\tau) = Y(\tau, .) \in \mathbb{X}^1$, rewrite equation (2.29) as

$$(2.33) \qquad (\partial_t - J)\mathcal{Y} = \varepsilon^{\alpha-1}[G_\alpha(\mathcal{Y}) - \partial_\tau\mathcal{Y} + R_\varepsilon(\mathcal{Y})]$$

and look for approximate solutions to (2.33) of the form

$$(2.34) \qquad \mathcal{Y} = \mathcal{Y}_0 + \varepsilon^{\alpha-1}\mathcal{Y}_1 + o(\varepsilon^{\alpha-1}),$$

where $\mathcal{Y}_j(\tau) \in \mathbb{X}^1$, $j = 0, 1$. Inserting expansion (2.34) in equation (2.33) yields at leading order in ε

$$(2.35) \qquad (\partial_t - J)\mathcal{Y}_0 = 0,$$

i.e. $\mathcal{Y}_0(\tau) \in \ker(\partial_t - J)$. Consequently, the principal part of the approximate solution takes the form

$$(2.36) \qquad (\mathcal{Y}_0(\tau))(t) = a(\tau)e^{it}e_1 + \bar{a}(\tau)e^{-it}e_{-1},$$

where $a(\tau) \in \ell_p(\mathbb{Z}, \mathbb{C})$. Similarly, identification at order $\varepsilon^{\alpha-1}$ yields

$$(2.37) \qquad (\partial_t - J)\mathcal{Y}_1 = G_\alpha(\mathcal{Y}_0) - \partial_\tau \mathcal{Y}_0.$$

According to lemma 2.5, this nonhomogeneous equation can be solved under the compatibility condition $P\left[G_\alpha(\mathcal{Y}_0) - \partial_\tau \mathcal{Y}_0\right] = 0$, i.e. $\mathcal{Y}_0$ must satisfy the amplitude equation

$$(2.38) \qquad \partial_\tau \mathcal{Y}_0 = P\,G_\alpha(\mathcal{Y}_0).$$

Then equation (2.37) becomes

$$(2.39) \qquad (\partial_t - J)\mathcal{Y}_1 = (I - P)\,G_\alpha(\mathcal{Y}_0),$$

which determines $\mathcal{Y}_1(\tau)$ as a function of $\mathcal{Y}_0(\tau)$, up to an element of $\ker(\partial_t - J)$. At our level of approximation, we can arbitrarily fix $\mathcal{Y}_1(\tau) \in \mathrm{range}(\partial_t - J)$, which yields according to lemma 2.5

$$(2.40) \qquad \mathcal{Y}_1(\tau) = \mathcal{K}\,G_\alpha(\mathcal{Y}_0(\tau)).$$

As a conclusion, we have obtained an approximate solution to equation (2.5),

$$(2.41) \qquad X_{\mathrm{app}}^\varepsilon(t) = \varepsilon[\mathcal{Y}_0(\varepsilon^{\alpha-1}t)](t) + \varepsilon^\alpha[\mathcal{Y}_1(\varepsilon^{\alpha-1}t)](t),$$

where $\mathcal{Y}_0$ denotes a solution to the amplitude equation (2.38) taking the form (2.36) and the corrector $\mathcal{Y}_1$ is defined by (2.40).

In section 3 we justify the above formal multiple-scale analysis by obtaining a suitable error bound on the approximate solutions. The overall strategy is based on the following ideas. In section 3.1, we check that the approximate solution (2.41) solves (2.5) up to an error that remains $\mathcal{O}(\varepsilon^{\alpha+\eta})$ for $t \in [0, t_m]$, with $t_m = T\,\varepsilon^{1-\alpha}$ ($T > 0$ fixed) and $\eta = \min(\alpha-1, \beta, \gamma) > 0$ (this corresponds to the orders of the terms of (2.33) neglected in the above analysis). From this result and Gronwall's inequality, we get the error estimate

$$\|X_{\mathrm{app}}^\varepsilon(t) - X(t)\|_{L^\infty([0,t_m],\ell_p^2)} = \mathcal{O}(t_m\,\varepsilon^{\alpha+\eta}),$$

provided $\|X_{\mathrm{app}}^\varepsilon(0) - X(0)\|_p = \mathcal{O}(t_m\,\varepsilon^{\alpha+\eta})$ (section 3.2). Consequently, if t_m were bounded then both terms of approximation (2.41) would be relevant for $t \in [0, t_m]$. However, in our case t_m diverges for $\varepsilon \to 0$, hence we get a larger error

$$\|X_{\mathrm{app}}^\varepsilon(t) - X(t)\|_{L^\infty([0,t_m],\ell_p^2)} = \mathcal{O}(\varepsilon^{1+\eta}).$$

As a result, only the lowest-order term of approximation (2.41) is relevant on $\mathcal{O}(\varepsilon^{1-\alpha})$ times, which finally yields the approximate solution to (2.5)

$$(2.42) \qquad X_a^\varepsilon(t) = \varepsilon[\mathcal{Y}_0(\varepsilon^{\alpha-1}t)](t) = \varepsilon a(\varepsilon^{\alpha-1}t)e^{it}e_1 + \varepsilon\bar{a}(\varepsilon^{\alpha-1}t)e^{-it}e_{-1}.$$

Let us examine more closely the amplitude equation satisfied by $a(\tau) \in \ell_p(\mathbb{Z}, \mathbb{C})$. The nonlinear term of (2.38) can be explicitly computed following the lines of [Jam11]; see the appendix for details. More precisely, we have

$$(2.43) \qquad\qquad i\partial_\tau a = \delta^+ f(\delta^- a),$$

where

$$(f(a))_n = \omega_0 \, a_n \, |a_n|^{\alpha-1},$$

$$\omega_0 = (k_- + k_+) \, 2^{\frac{\alpha-3}{2}} \, \frac{\alpha \, \Gamma(\frac{\alpha}{2})}{\sqrt{\pi}(\alpha+1)\Gamma(\frac{\alpha+1}{2})}$$

and $\Gamma(x) = \int_0^{+\infty} e^{-t} \, t^{x-1} \, dt$ denotes Euler's Gamma function. Equation (2.43) reads component-wise

$$(2.44) \qquad\qquad i\frac{\partial a_n}{\partial \tau} = \omega_0 \, (\Delta_{\alpha+1} a)_n, \quad n \in \mathbb{Z},$$

where the nonlinear difference operator

$$(\Delta_{\alpha+1} a)_n = (a_{n+1} - a_n) \, |a_{n+1} - a_n|^{\alpha-1} - (a_n - a_{n-1}) \, |a_n - a_{n-1}|^{\alpha-1}$$

is the discrete $(\alpha+1)$-Laplacian.

2.5. Qualitative properties of the amplitude equation. In this section we establish the well-posedness of the differential equation (2.44), point out some invariances and conserved quantities yielding global existence results, and study the existence of spatially localized solutions which do not decay when $\tau \to +\infty$.

2.5.1. Conserved quantities and well-posedness.
Let $a \in \mathcal{C}^1((T_{\min}, T_{\max}), \ell_{\alpha+1}(\mathbb{Z}, \mathbb{C}))$ denote a solution to (2.43) defined on some time interval $(T_{\min}, T_{\max})$. One can readily check that the quantity

$$\|\delta^+ a\|_{\alpha+1}^{\alpha+1} = \sum_{n \in \mathbb{Z}} |a_{n+1} - a_n|^{\alpha+1}$$

is conserved along evolution. This property is linked with the Hamiltonian structure of equation (2.44), which can be formally written

$$(2.45) \qquad \frac{\partial a_n}{\partial \tau} = i\frac{\partial \mathcal{H}}{\partial \bar{a}_n}, \quad n \in \mathbb{Z}, \quad \text{with} \quad \mathcal{H} = \frac{2\omega_0}{\alpha+1} \, \|\delta^+ a\|_{\alpha+1}^{\alpha+1}.$$

More precisely, setting

$$a = \pi_1 \begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{\sqrt{2}}(q - ip),$$

the solutions $a \in \mathcal{C}^1((T_{\min}, T_{\max}), \ell_2(\mathbb{Z}, \mathbb{C}))$ of (2.43) correspond to solutions $(q, p) \in \mathcal{C}^1((T_{\min}, T_{\max}), \ell_2^2)$ of the Hamiltonian system

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J \, \nabla\mathcal{H}(q, p),$$

where

$$\mathcal{H}(q,p) = \frac{2\omega_0}{\alpha+1} \sum_{n\in\mathbb{Z}} \left[\left(\frac{q_{n+1}-q_n}{\sqrt{2}} \right)^2 + \left(\frac{p_{n+1}-p_n}{\sqrt{2}} \right)^2 \right]^{\frac{\alpha+1}{2}}$$

is defined on the real Hilbert space ℓ_2^2.

Equation (2.44) admits the gauge invariance $a_n \to a_n\, e^{i\varphi}$, the translational invariance $a_n \to a_n + c$ and a scale invariance, since any solution $(a_n)_{n\in\mathbb{Z}}$ of (2.44) generates a one-parameter family of solutions $\left(\varepsilon\, a_n(|\varepsilon|^{\alpha-1}\,\tau) \right)_{n\in\mathbb{Z}}$, $\varepsilon \in \mathbb{R}$. Several conserved quantities of (2.44) can be associated to these invariances via Noether's theorem. The scale invariance and the invariance by time translation correspond to the conservation of $\mathcal{H}$. The gauge invariance yields the conserved quantity

$$\|a\|_2^2 = \sum_{n\in\mathbb{Z}} |a_n|^2$$

whenever $a \in \mathcal{C}^1((T_{\min}, T_{\max}), \ell_2(\mathbb{Z}, \mathbb{C}))$. In the same way, the translational invariance yields the additional conserved quantity

$$\mathcal{P} = \sum_{n\in\mathbb{Z}} a_n$$

provided $a \in \mathcal{C}^1((T_{\min}, T_{\max}), \ell_1(\mathbb{Z}, \mathbb{C}))$.

In the sequel we use the notation $\ell_p = \ell_p(\mathbb{Z}, \mathbb{C})$. The following lemma ensures the local well-posedness of equation (2.44) in ℓ_p for $p \in [1, +\infty]$ and its global well-posedness for $p \in [1, 1+\alpha]$ (in particular for $p = 2$).

Lemma 2.7. *Let $a^0 \in \ell_p$. Equation (2.44) with initial data $a(0) = a^0$ admits a unique solution $a \in \mathcal{C}^2((T_{\min}, T_{\max}), \ell_p)$, defined on a maximal interval of existence $(T_{\min}, T_{\max})$ depending a priori on a^0 (with $T_{\min} < 0 < T_{\max}$). One has in addition*

(2.46) $$T_{\max} \geq T_1\, \|a^0\|_p^{1-\alpha}, \quad T_{\min} \leq -T_1\, \|a^0\|_p^{1-\alpha},$$

with $T_1 = [(\alpha-1)\, 2^{\alpha+1} \omega_0]^{-1}$. Moreover, $(T_{\min}, T_{\max}) = \mathbb{R}$ if $p \in [1, \alpha+1]$.

Proof. Since $\alpha > 1$ we have $\Delta_{\alpha+1} \in \mathcal{C}^1(\ell_p, \ell_p)$ and thus the Cauchy problem for (2.44) is locally well-posed in ℓ_p. Therefore, for all initial condition $a^0 \in \ell_p$, equation (2.44) admits a unique maximal solution $a \in \mathcal{C}^1((T_{\min}, T_{\max}), \ell_p)$, and a bootstrap argument yields then $a \in \mathcal{C}^2((T_{\min}, T_{\max}), \ell_p)$.

To prove estimates (2.46), we rewrite (2.44) in the form

(2.47) $$ia(\tau) = ia^0 + \delta^+ \int_0^\tau f(\delta^- a(s))\, \mathrm{d}s.$$

Since $\|f(a)\|_p \leq \omega_0 \|a\|_p^\alpha$, we get

$$\|a(\tau)\|_p \leq \|a^0\|_p + 2^{\alpha+1}\, \omega_0 \int_0^\tau \|a(s)\|_p^\alpha\, \mathrm{d}s, \quad 0 \leq \tau < T_{\max}.$$

As in the proof of lemma 2.6, a Gronwall-type estimate yields then

(2.48) $$\|a(\tau)\|_p \leq \varrho_{\varepsilon,\lambda}(\tau), \quad 0 \leq \tau < \varepsilon^{1-\alpha}\, T_1,$$

for the parameter choice $\varepsilon = \|a^0\|_p$ and $\lambda = 2^{\alpha+1}\,\omega_0$, where $\varrho_{\varepsilon,\lambda}(\tau)$ is the solution to the differential equation (2.24) with explicit form (2.25) defined up to $\tau = \varepsilon^{1-\alpha}\,T_1$. Consequently, bound (2.48) yields the first estimate of (2.46), and the estimate for $\tau \leq 0$ is the same as for $\tau \geq 0$ owing to the invariance $a(\tau) \to \bar{a}(-\tau)$ of (2.44).

Global well-posedness in ℓ_p for $p \in [1, \alpha+1]$ follows from the fact that $\Delta_{\alpha+1}$, $D\Delta_{\alpha+1}$ are bounded on bounded sets in ℓ_p and $\|a(\tau)\|_p$ is bounded on bounded time intervals. To prove this second property, we deduce from (2.47)

$$
\begin{aligned}
\|a(\tau)\|_p &\leq \|a^0\|_p + 2\,\omega_0 \int_0^\tau \|\delta^- a(s)\|_p \, \|\delta^- a(s)\|_\infty^{\alpha-1} \, \mathrm{d}s, \\
&\leq \|a^0\|_p + 2\,\omega_0 \int_0^\tau \|\delta^- a(s)\|_p \, \|\delta^- a(s)\|_{\alpha+1}^{\alpha-1} \, \mathrm{d}s, \\
&\leq \|a^0\|_p + 4\,\omega_0 \, \|\delta^- a^0\|_{\alpha+1}^{\alpha-1} \int_0^\tau \|a(s)\|_p \, \mathrm{d}s,
\end{aligned}
$$

where we have used the fact that $\|f(a)\|_p \leq \omega_0 \, \|a\|_p \, \|a\|_\infty^{\alpha-1}$ and $\|\delta^- a(\tau)\|_{\alpha+1}$ is conserved. Now we have by Gronwall's lemma

$$
(2.49) \qquad \|a(\tau)\|_p \leq e^{\sigma\,\tau}\,\|a^0\|_p, \quad \sigma = 2^{\alpha+1}\,\omega_0\,\|a^0\|_{\alpha+1}^{\alpha-1},
$$

and the proof is complete. $\qquad\square$

Remark 2.8. The bound (2.49) can be improved for $p = 2$ and $p = \alpha + 1$, since $\|a(\tau)\|_2 = \|a^0\|_2$ is bounded and a sharper estimate can be deduced from (2.47) :

$$
\|a(\tau)\|_{\alpha+1} \leq \|a^0\|_{\alpha+1} + 2\omega_0\,\tau\,\|\delta^- a^0\|_{\alpha+1}^\alpha.
$$

2.5.2. Spatially localized solutions. Another important feature of equation (2.44) is the absence of scattering for square-summable solutions. More precisely, the following result ensures that all (nontrivial) solutions to (2.44) in ℓ_2 satisfy $\inf_{\tau \in \mathbb{R}} \|a(\tau)\|_\infty > 0$, which implies that they do not completely disperse. The proof is based on the conservation of ℓ_2 norm and energy, an idea introduced in [KKFA08] in the context of the disordered discrete nonlinear Schrödinger equation.

Lemma 2.9. *Let $a^0 \in \ell_2$ with $a^0 \neq 0$ and $a \in \mathcal{C}^1(\mathbb{R}, \ell_2)$ denote the solution to (2.44) with $a(0) = a^0$. Then we have*

$$
\forall \tau \in \mathbb{R}, \quad \|a(\tau)\|_\infty \geq \left(\frac{\|\frac{1}{2}\delta^+ a^0\|_{\alpha+1}^{\alpha+1}}{\|a^0\|_2^2} \right)^{\frac{1}{\alpha-1}}.
$$

Proof. Simply use the conserved quantities $\|\delta^+ a\|_{\alpha+1}$ and $\|a\|_2$ from section 2.5.1, and estimate thanks to the triangle and interpolation inequalities:

$$
\begin{aligned}
\|\delta^+ a^0\|_{\alpha+1} &= \|\delta^+ a(\tau)\|_{\alpha+1} \\
&\leq 2\|a(\tau)\|_{\alpha+1} \\
&\leq 2\|a(\tau)\|_\infty^{1-\frac{2}{\alpha+1}} \|a(\tau)\|_2^{\frac{2}{\alpha+1}} = 2\|a(\tau)\|_\infty^{1-\frac{2}{\alpha+1}} \|a^0\|_2^{\frac{2}{\alpha+1}}.
\end{aligned}
$$

$\qquad\square$

When a^0 is restricted to some subset $\mathcal{C}$ of ℓ_2, the following result provides a simpler estimate involving the constant

$$
(2.50) \qquad \mathcal{I}(\mathcal{C}) = \inf \{ Q_\alpha(a), \, a \in \mathcal{C}, \, a \neq 0 \},
$$

where

$$(2.51) \qquad Q_\alpha(a) = \frac{\|\delta^+ a\|_{\alpha+1}}{\|a\|_2}.$$

Corollary 2.10. *Keep the notations of lemma 2.9 and assume $a^0 \in \mathcal{C} \subset \ell_2$. Then we have*

$$(2.52) \qquad \forall \tau \in \mathbb{R}, \quad \|a(\tau)\|_\infty \geq \left(\frac{1}{2} \mathcal{I}(\mathcal{C}) \right)^{\frac{\alpha+1}{\alpha-1}} \|a^0\|_2.$$

Proof. Use lemma 2.9 and the identity

$$\left(\frac{\|\frac{1}{2}\delta^+ a^0\|_{\alpha+1}^{\alpha+1}}{\|a^0\|_2^2} \right)^{\frac{1}{\alpha-1}} = \|a^0\|_2 \left(\frac{1}{2} Q_\alpha(a^0) \right)^{\frac{\alpha+1}{\alpha-1}}.$$

$\square$

Remark 2.11. The above result is useless for $\mathcal{C} = \ell_2$ since $\mathcal{I}(\ell_2) = 0$. Indeed, considering the sequence $a^N = \mathbb{1}_{\{1,\dots,N\}}$ (with $\mathbb{1}$ denoting the indicator function) one can check that $Q_\alpha(a^N) = 2^{\frac{1}{\alpha+1}} N^{-1/2} \to 0$ as $N \to +\infty$.

Remark 2.12. If $\mathcal{C}$ is a finite-dimensional linear subspace of ℓ_2, then $\mathcal{I}(\mathcal{C}) > 0$ (Q_α is the ratio of two equivalent norms on $\mathcal{C}$). Moreover, if $\mathcal{I}(\mathcal{C}) > 0$ on some subspace $\mathcal{C}$ of ℓ_2 then the norms $\| \|_\infty$ and $\| \|_2$ are equivalent on $\mathcal{C}$ (this follows from the case $\tau = 0$ of (2.52) and the continuous embedding $\ell_2 \subset \ell_\infty$).

Lemma 2.9 and corollary 2.10 show that all square-summable localized solutions do not decay as $\tau \to \pm\infty$. One of the simplest type of localized solutions to (2.44) corresponds to time-periodic oscillations (discrete breathers), which have been studied in a number of works (see [JS13] and references therein). Equation (2.44) admits time-periodic solutions of the form

$$(2.53) \qquad a_n(\tau) = \varepsilon \, v_n \, e^{i \, \omega_0 \, |\varepsilon|^{\alpha-1} \tau},$$

where $v = (v_n)_{n\in\mathbb{Z}}$ is a real sequence and $\varepsilon \in \mathbb{R}$ an arbitrary constant, if and only if v satisfies

$$(2.54) \qquad v_n = -(\Delta_{\alpha+1}v)_n, \quad n \in \mathbb{Z}.$$

In particular, nontrivial solutions to (2.54) satisfying $\lim_{n\to\pm\infty} v_n = 0$ correspond to breather solutions to (2.44) given by (2.53). The following existence theorem for spatially symmetric breathers has been proved in [JS13] using a reformulation of (2.54) as a two-dimensional mapping.

Theorem 2.13. *The stationary DpS equation (2.54) admits solutions v_n^i ($i = 1, 2$) satisfying*

$$\lim_{n\to\pm\infty} v_n^i = 0,$$

$$(-1)^n \, v_n^i > 0, \quad |v_n^i| > |v_{n-1}^i| \quad \text{for all } n \leq 0,$$

$$\text{and} \quad v_n^1 = v_{-n}^1, \quad v_n^2 = -v_{-n+1}^2, \quad \text{for all } n \in \mathbb{Z}.$$

Furthermore, for all $q \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that the above-mentioned solutions v_n^i satisfy, for $i = 1, 2$:

$$\forall n \geq n_0, \quad |v_n^i| \leq q^{1+\alpha^{n-n_0}}.$$

Remark 2.14. These solutions are thus doubly exponentially decaying, so that they belong to ℓ_p for all $p \in [1, \infty]$.

One may wonder if results analogous to lemma 2.9 and theorem 2.13 hold true for the original lattice (2.4). The proofs of the above results heavily rely on the gauge invariance of (2.44) which implies the conservation of $\|a(\tau)\|_2$. Such properties are not available for system (2.4), hence the same methodology cannot be directly applied to Newton's cradle. These problems will be solved in the next section through the justification of approximation (2.42) on long time scales.

2.6. Error bounds and applications. In this section, we give several error bounds in order to justify the expansions of section 2.4, for small amplitude solutions and long (but finite) time intervals. From these error bounds, we also infer stability results for long-lived breather solutions to the original lattice model, as well as lower bounds for the amplitudes of small solutions valid over long times (see section 2.6.3).

2.6.1. Asymptotics for times $\mathcal{O}(1/\varepsilon^{\alpha-1})$. In theorem 2.15 below, one considers any solution $a \in \mathcal{C}^2([0,T], \ell_p)$ to equation (2.43) and constructs a family X_a^ε of approximate solutions to (2.5), whose amplitudes are $\mathcal{O}(\varepsilon)$ and determined by a and ε. These approximate solutions are $\mathcal{O}(\varepsilon^{1+\eta})$-close to exact solutions for some constant $\eta > 0$ specified below and $t \in [0, T/\varepsilon^{\alpha-1}]$. The proof of theorem 2.15 is detailed in section 3.2.

Theorem 2.15. *Let $\eta = \min(\alpha - 1, \beta, \gamma)$ and fix two constants $C_i, T > 0$. There exist a c.n.i.f. $\varepsilon_T > 0$ and a c.n.d.f. $C_T \geq C_i$ such that the following holds:*
For all solution $a \in \mathcal{C}^2([0,T], \ell_p)$ to equation (2.43) with $N \stackrel{def}{=} \|a\|_{L^\infty([0,T], \ell_p)}$ and for all $\varepsilon \leq \varepsilon_T(N)$, we define

$$(2.55) \qquad X_a^\varepsilon(t) = \frac{\varepsilon}{\sqrt{2}} a(\varepsilon^{\alpha-1} t) \, e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + \text{c.c.}$$

Then, for all $X^0 \in \ell_p^2$ satisfying

$$\|X^0 - X_a^\varepsilon(0)\|_p \leq C_i \varepsilon^{1+\eta},$$

the solution $X(t) \in \ell_p^2$ of equation (2.5) with $X(0) = X^0$ is defined at least for $t \in [0, T/\varepsilon^{\alpha-1}]$ and satisfies

$$(2.56) \qquad \|X(t) - X_a^\varepsilon(t)\|_p \leq C_T(N) \, \varepsilon^{1+\eta} \text{ for all } t \in [0, T/\varepsilon^{\alpha-1}].$$

Remark 2.16. The case $\phi = 0$ of equation (2.4) (harmonic on-site potential Φ) corresponds to fixing $\gamma = +\infty$, which yields $\eta = \min(\alpha - 1, \beta)$. Similarly, the case $W = 0$ (pure Hertzian-type interaction potential $V = V_\alpha$) is obtained with $\beta = +\infty$ and $\eta = \min(\alpha - 1, \gamma)$.

Remark 2.17. By lemma 2.6, the solution X of equation (2.5) satisfies $X = (x, \dot{x})^T \in \mathcal{C}^2([0, T/\varepsilon^{\alpha-1}], \ell_p^2)$, hence $x \in \mathcal{C}^3([0, T/\varepsilon^{\alpha-1}], \ell_p)$. Moreover, it follows from lemma 2.7 that $X_a^\varepsilon = (x_a^\varepsilon, y_a^\varepsilon)^T \in \mathcal{C}^2([0, T/\varepsilon^{\alpha-1}], \ell_p^2)$, hence $x_a^\varepsilon \in \mathcal{C}^2([0, T/\varepsilon^{\alpha-1}], \ell_p)$. Consequently, the exact solution x of (2.4) is generally more regular than its approximation x_a^ε, which is an unusual property in the context of modulation equations.

Example 2.18. In the case $\alpha = 3/2$ (as for the classical Hertz force), if W and ϕ are $\mathcal{C}^3$ at both sides of the origin, then $\beta, \gamma \geq 1/2$ in (2.2) and (2.3) and we have by theorem 2.15

$$\|X(t) - X_a^\varepsilon(t)\|_p \leq C_T(N) \, \varepsilon^{3/2} \text{ for all } t \in [0, T \varepsilon^{-1/2}].$$

As a corollary of theorem 2.15 and previous estimates on the solutions to the amplitude equation (2.43), one obtains theorem 2.19 below. Roughly speaking, for *all* sufficiently small initial data $X^0 \in \ell_p$, this theorem provides an approximation X_A of the solution to (2.5) (with amplitude given by a solution to (2.43)) valid on $\mathcal{O}(\|X^0\|_p^{1-\alpha})$ time scales.

Theorem 2.19. *Let $\eta = \min(\alpha - 1, \beta, \gamma)$ and $C_i > 0$. There exists $T_m \in (0, +\infty]$ such that for all $T \in (0, T_m)$, there exist $\tilde{\varepsilon}_T > 0$ and $\tilde{C}_T \geq C_i$ such that the following properties hold. For all $X^* = (x^*, y^*)^T \in \ell_p^2$ such that $0 < \varepsilon \stackrel{def}{=} \|X^*\|_p \leq \tilde{\varepsilon}_T$, for all $X^0 \in \ell_p^2$ satisfying $\|X^0 - X^*\|_p \leq C_i \varepsilon^{1+\eta}$, the solution $X(t) \in \ell_p^2$ to equation (2.5) with $X(0) = X^0$ is defined at least for $t \in [0, T\varepsilon^{1-\alpha}]$. This solution satisfies*

$$(2.57) \qquad \|X(t) - X_A(t)\|_p \leq \tilde{C}_T \varepsilon^{1+\eta} \text{ for all } t \in [0, T\varepsilon^{1-\alpha}],$$

where

$$X_A(t) = \frac{1}{\sqrt{2}} A(t) \, e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + \text{c.c.,}$$

and $A \in \mathcal{C}^2([0, T\varepsilon^{1-\alpha}], \ell_p)$ is the solution to equation (2.43) with $\mathcal{O}(\varepsilon)$ initial condition $A(0) = (x^* - i\, y^*)/\sqrt{2}$. Moreover, if $p \in [1, 1+\alpha]$ then $T_m = +\infty$.

Proof. By lemma 2.7, there exists a unique maximal solution A of (2.43) with initial condition $A(0)$ defined as above from X^*. Due to the scale invariance of (2.43), the initial condition $a(0) = \varepsilon^{-1} A(0)$ yields the solution $a(\tau) = \varepsilon^{-1} A(\varepsilon^{1-\alpha} \tau)$. Since $\|a(0)\|_p = \sqrt{2}/2$ by construction, lemma 2.7 ensures that $a(\tau)$ is defined and bounded in ℓ_p for $\tau \in [0, T]$ whenever $T < T_1 2^{(\alpha-1)/2}$. In addition this property is true for all $T \in (0, +\infty)$ when $p \in [1, 1+\alpha]$. This leads us to define $T_m = T_1 2^{(\alpha-1)/2}$ for $p > 1 + \alpha$ and $T_m = +\infty$ for $p \in [1, 1+\alpha]$.

Now let us consider $T < T_m$ being fixed, so that $A(t)$ is defined and bounded in ℓ_p for $t \in [0, T\varepsilon^{1-\alpha}]$. Using either bounds (2.48) or (2.49) (the latter being valid for $p \in [1, 1+\alpha]$), there exists $M_T > 0$ independent of X^* such that $N \stackrel{def}{=} \|a\|_{L^\infty([0,T], \ell_p)} \leq M_T$.

With the above remarks one can apply theorem 2.15 for $\varepsilon \leq \varepsilon_T(M_T) = \tilde{\varepsilon}_T$. Noticing that $X_A(t) = X_a^\varepsilon(t)$, one obtains estimate (2.57) with $\tilde{C}_T = C_T(M_T)$, which proves theorem 2.19. $\qquad\square$

2.6.2. Asymptotics for times $\mathcal{O}(|\ln \varepsilon| \varepsilon^{1-\alpha})$. Theorem 2.20 below provides a different kind of error estimate, where the multiple-scale approximation is controlled on longer $\mathcal{O}(|\ln \varepsilon| \varepsilon^{1-\alpha})$ time scales, at the expense of lowering the precision of (2.56). These estimates are valid when the Ansatz $X_a^\varepsilon(t) \in \ell_p^2$ is bounded for $t \in \mathbb{R}^+$, i.e. X_a^ε is constructed from a solution $a \in L^\infty(\mathbb{R}^+, \ell_p)$ of (2.43). The proof of this result is detailed in section 3.3.

Theorem 2.20. *Let $\eta = \min(\alpha - 1, \beta, \gamma)$, $\mu \in (0, \eta)$ and $C_i > 0$. Fix $a \in L^\infty(\mathbb{R}^+, \ell_p)$ solution to equation (2.43) with $\|a\|_{L^\infty(\mathbb{R}^+, \ell_p)} = N$ and consider*

$$X_a^\varepsilon(t) = \frac{\varepsilon}{\sqrt{2}} a(\varepsilon^{\alpha-1} t) e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + \text{c.c.}$$

There exist positive constants $\varepsilon_0(\mu, C_i, N)$, $C_1(C_i, N)$ and a c.n.i.f. $\nu(N)$ such that for all $\varepsilon \leq \varepsilon_0$, if $X^0 \in \ell_p^2$ satisfies $\|X^0 - X_a^\varepsilon(0)\|_p \leq C_i |\ln \varepsilon| \varepsilon^{1+\eta}$, then the solution $X(t) \in \ell_p^2$ to equation (2.5) with $X(0) = X^0$ is defined for $t \in [0, \mu \nu |\ln \varepsilon| \varepsilon^{1-\alpha}]$ and satisfies

$$(2.58) \qquad \|X(t) - X_a^\varepsilon(t)\|_p \leq C_1 |\ln \varepsilon| \varepsilon^{1+\eta-\mu}, \quad t \in \left[0, \mu \nu |\ln \varepsilon| \varepsilon^{1-\alpha}\right].$$

In the same way as theorem 2.19 was deduced from theorem 2.15, theorem 2.21 below follows directly from theorem 2.20. The proof requires all solutions to (2.43) to be global and bounded in ℓ_p (due to the same assumption made on a in theorem 2.20), hence we have to restrict to $p = 2$.

Theorem 2.21. *Let $\eta = \min(\alpha - 1, \beta, \gamma)$, $\mu \in (0, \eta)$ and $C_i > 0$. There exist positive constants $\varepsilon_0(\mu, C_i)$, $C_l(C_i)$ and ν such that the following holds. For any $X^* = (x^*, y^*)^T \in \ell_2^2$ such that $\varepsilon \overset{def}{=} \|X^*\|_2 \leq \varepsilon_0$, we consider the solution $A \in \mathcal{C}^2(\mathbb{R}, \ell_2)$ to equation (2.43) with $\mathcal{O}(\varepsilon)$ initial condition $A(0) = (x^* - i\,y^*)/\sqrt{2}$, and we define*

$$X_A(t) = \frac{1}{\sqrt{2}} A(t)\, e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + \text{c.c.}$$

Then, for all $X^0 \in \ell_2^2$ satisfying $\|X^0 - X^*\|_2 \leq C_i\,|\ln\varepsilon|\varepsilon^{1+\eta}$, the solution $X(t) \in \ell_p^2$ to equation (2.5) with $X(0) = X^0$ is defined for $t \in [0, \mu\,\nu\,|\ln\varepsilon|\,\varepsilon^{1-\alpha}]$ and satisfies

$$(2.59) \qquad \|X(t) - X_A(t)\|_2 \leq C_l\,|\ln\varepsilon|\,\varepsilon^{1+\eta-\mu}, \quad t \in \left[0, \mu\,\nu\,|\ln\varepsilon|\,\varepsilon^{1-\alpha}\right].$$

Proof. We proceed exactly as in the proof of theorem 2.19 for $p = 2$, except the solutions A, a of (2.43) are now global in time, $N \overset{def}{=} \|a\|_{L^\infty(\mathbb{R}^+, \ell_2)} = \|a(0)\|_2 = \sqrt{2}/2$, and we use theorem 2.20 instead of theorem 2.15. ☐

2.6.3. Long-lived localized solutions. We can apply theorem 2.13 to generate breather solutions to the amplitude DpS equation (2.44) which provide approximate solutions for theorems 2.15, 2.19, 2.20 and 2.21. Hence, we obtain stable exact solutions to the original nonlinear lattice (2.4), close to breathers, over the corresponding time scales.

Theorem 2.22. *Let $\eta = \min(\alpha - 1, \beta, \gamma)$ and fix two constants $C_i, T > 0$. Consider a solution $v^i = (v_n^i)_{n \in \mathbb{Z}}$ ($i = 1, 2$) of the stationary DpS equation (2.54) described in theorem 2.13. There exist $\varepsilon_T, C_T > 0$ such that for all $\varepsilon \in (0, \varepsilon_T]$, for all $X^0 \in \ell_p^2$ satisfying*

$$(2.60) \qquad \|X^0 - (\sqrt{2}\varepsilon v^i, 0)^T\|_p \leq C_i \varepsilon^{1+\eta},$$

the solution $X(t) \in \ell_p^2$ to equation (2.5) with $X(0) = X^0$ is defined at least for $t \in [0, T\varepsilon^{1-\alpha}]$ and satisfies

$$(2.61) \qquad \|X(t) - X_b^\varepsilon(t)\|_p \leq C_T \varepsilon^{1+\eta} \text{ for all } t \in [0, T\varepsilon^{1-\alpha}],$$

where

$$X_b^\varepsilon(t) = \sqrt{2}\,\varepsilon\,(v^i \cos(\Omega t), -v^i \sin(\Omega t))^T, \quad \Omega = 1 + \omega_0\,\varepsilon^{\alpha-1}.$$

Proof. Consider the breather solution of (2.44) given by (2.53) with $v = v^i$, $\varepsilon = 1$, and apply theorem 2.15. ☐

As a result of estimate (2.61), the initial condition $X(0) = (\sqrt{2}\varepsilon v^i, 0)^T$ generates long-lived breather solutions $\tilde{X}_b^\varepsilon$ defined for $t \in [0, T\,\varepsilon^{1-\alpha}]$ and taking the form $\tilde{X}_b^\varepsilon(t) = X_b^\varepsilon(t) + \mathcal{O}(\varepsilon^{1+\eta})$. These solutions are stable in ℓ_p^2 on the corresponding time scale since condition (2.60) implies

$$\|X(t) - \tilde{X}_b^\varepsilon(t)\|_p \leq 2\,C_T\varepsilon^{1+\eta} \text{ for all } t \in [0, T\varepsilon^{1-\alpha}]$$

(this follows by using (2.61) and the triangle inequality).

Using theorem 2.21 and corollary 2.10, we also obtain lower bounds for the ampli-tudes of small localized solutions over long times. This result is valid for all initial data in subsets $\mathcal{C}$ of ℓ_2^2 such that $\mathcal{I}(\mathcal{C}) > 0$, where $\mathcal{I}(\mathcal{C})$ is defined as in (2.50)-(2.51) with the choice of norms (2.6) (for which the canonical isomorphism between $(\ell_p(\mathbb{Z}, \mathbb{R}))^2$ and $\ell_p(\mathbb{Z}, \mathbb{C})$ is an isometry). As already noticed in remark 2.12, one has $\mathcal{I}(\mathcal{C}) > 0$ whenever $\mathcal{C}$ is a finite-dimensional linear subspace of ℓ_2^2.

Proposition 2.23. *Keep the notations of theorem 2.21. Let $\mathcal{C}$ denote a subset of ℓ_2^2 such that $\mathcal{I}(\mathcal{C}) > 0$. There exists $\varepsilon_1(\mu, C_i, \mathcal{I}(\mathcal{C})) > 0$ such that for all $\varepsilon \in (0, \varepsilon_1]$ and $X^0 \in \mathcal{C}$ with $\|X^0\|_2 = \varepsilon$, the solution X to equation (2.5) given by theorem 2.21 is bounded from below, namely:*

$$(2.62) \qquad \forall t \in \left[0, \mu \, \nu \, |\ln \varepsilon| \, \varepsilon^{1-\alpha}\right], \quad \|X(t)\|_\infty \geq M \, \varepsilon$$

with $M = \frac{1}{2} \left(\frac{1}{2} \mathcal{I}(\mathcal{C})\right)^{\frac{\alpha+1}{\alpha-1}}$.

Proof. Fix $X^0 = X^*$ in theorem 2.21 and note that

$$\|X(t)\|_\infty \geq \|X_A(t)\|_\infty - \|X(t) - X_A(t)\|_\infty$$
$$\geq \|X_A(t)\|_\infty - \|X(t) - X_A(t)\|_2.$$

Now, estimating $\|X_A(t)\|_\infty$ thanks to corollary 2.10 and $\|X(t) - X_A(t)\|_2$ with theorem 2.21 gives $\|X(t)\|_\infty \geq \varepsilon \left(2M - C_l \, |\ln \varepsilon| \, \varepsilon^{\eta-\mu}\right)$. Since $\eta > \mu$, this estimate implies (2.62) provided ε is small enough. $\square$

Example 2.24. Consider solutions $x = (x_n)_{n \in \mathbb{Z}}$ to (2.4) with unperturbed initial po-sitions, and with a group of N consecutive particles having the same initial velocity v_i ($N \geq 1$ being fixed). This corresponds to fixing

$$\forall n \in \mathbb{Z}, \ x_n(0) = 0, \quad \dot{x}_n(0) = \begin{cases} v_i & \text{if } n \in \{1, \dots, N\}, \\ 0 & \text{elsewhere,} \end{cases}$$

i.e. $X^0 = v_i \, (0, \mathbb{1}_{\{1,\dots,N\}})^T$. For $\mathcal{C} = \text{Span}\left((0, \mathbb{1}_{\{1,\dots,N\}})^T\right)$ one has

$$\mathcal{I}(\mathcal{C}) = Q_\alpha(\mathbb{1}_{\{1,\dots,N\}}) = 2^{\frac{1}{\alpha+1}} N^{-1/2} > 0$$

(see definition (2.51)). Consequently one can apply proposition 2.23, where $M = 2^{\frac{2\alpha-1}{1-\alpha}} N^{\frac{\alpha+1}{2(1-\alpha)}}$ and $\|X^0\|_2 = |v_i| \, N^{1/2} = \varepsilon$. This yields for all $\varepsilon \in (0, \varepsilon_1]$ and $t \in \left[0, \mu \, \nu \, |\ln \varepsilon| \, \varepsilon^{1-\alpha}\right]$:

$$\sup_{n \in \mathbb{Z}} (x_n^2(t) + \dot{x}_n^2(t))^{1/2} \geq 2^{\frac{2\alpha-1}{1-\alpha}} N^{\frac{1}{1-\alpha}} |v_i|.$$

To interpret estimate (2.62), it is interesting to recall that $\|X(t)\|_\infty$ is conserved along evolution for solutions to the linearized equation (2.12) (see remark 2.2), hence the bound (2.62) estimates the maximal decay of $\|X\|_\infty$ that could occur over long times due to purely nonlinear effects. This estimate can be compared with the classical Gronwall estimate given below.

Lemma 2.25. *Keep the notations of lemma 2.6. There exists a constant $\tilde{\varepsilon}_1 > 0$ such that the following property holds true. For all $\delta \in (0, 1)$, there exists $\tilde{T}(\delta) \in (0, T_0]$ such that for all $X^0 \in \ell_\infty^2$ with $\|X^0\|_\infty \leq \tilde{\varepsilon}_1$ one has*

$$(2.63) \qquad \|X(t)\|_\infty \geq \delta \, \|X^0\|_\infty, \quad \forall t \in [0, \tilde{T}(\delta) \, \|X^0\|_\infty^{1-\alpha}].$$

Proof. Using the triangle inequality in the Duhamel form (2.21), the time-invariance of $\|e^{Jt}X^0\|_\infty$ and estimate (2.22), one finds

$$\|X(t)\|_\infty \geq \|X^0\|_\infty - \lambda \int_0^t \|X(s)\|_\infty^\alpha \, \mathrm{d}s.$$

Then we deduce from the Gronwall estimate (2.27)

$$\|X(t)\|_\infty \geq \|X^0\|_\infty - t \, \lambda \, \theta^{\frac{\alpha}{1-\alpha}} \|X^0\|_\infty^\alpha,$$

from which the result follows easily. $\qquad\square$

Estimate (2.63) differs from (2.62) in the sense that it involves only the ℓ_∞ norm and holds true for all sufficiently small initial data in ℓ_∞^2. In addition it is valid on time scales of order $\|X^0\|_\infty^{1-\alpha}$, whereas (2.62) holds true on longer time scales of order $|\ln(\|X^0\|_2)| \, \|X^0\|_2^{1-\alpha}$.

3. Error bounds via Gronwall estimates. In this section we prove theorems 2.15 and 2.20. For this purpose, we check in section 3.1 the consistency of the Ansatz $X_{\mathrm{app}}^\varepsilon$ defined by (2.41) and conclude using Gronwall estimates (sections 3.2 and 3.3).

3.1. Estimate of the residual. In the sequel we consider solutions $a(\tau)$ to equation (2.44) such that $a \in L^\infty(I, \ell_p)$ for some closed time interval I. By lemma 2.7, this can be achieved for all initial condition $a(0) = a^0 \in \ell_p(\mathbb{Z}, \mathbb{C})$ by choosing $I = [0, T] \subset (T_{\min}, T_{\max})$ in the general case, or $I = [0, +\infty)$ in the particular case $p = 2$. The amplitude a determines the approximate solution $X_{\mathrm{app}}^\varepsilon$ to equation (2.5) introduced in section 2.4. Following equation (2.41), we recall that

$$X_{\mathrm{app}}^\varepsilon(t) = \varepsilon Y^\varepsilon(\tau, t), \quad \tau = \varepsilon^{\alpha-1}t,$$

where $Y^\varepsilon = Y_0 + \varepsilon^{\alpha-1}Y_1$ and $Y_k(\tau, t) = (\mathcal{Y}_k(\tau))(t)$ for $k = 0, 1$.

In this section we check that $X_{\mathrm{app}}^\varepsilon$ solves equation (2.5) up to the residual

$$(3.1) \qquad E^\varepsilon = \dot{X}_{\mathrm{app}}^\varepsilon - J\,X_{\mathrm{app}}^\varepsilon - G(X_{\mathrm{app}}^\varepsilon)$$

that remains $o(\varepsilon^\alpha)$ when $\varepsilon^{\alpha-1}t \in I$.

To this aim, we first prove the following lemma providing bounds on the approximate solutions $\mathcal{Y}_0$ and the correctors $\mathcal{Y}_1$ derived from the amplitudes $a(\tau)$. Below we denote by $\mathcal{C}_b^k(I, \mathbb{X}^1)$ the Banach space of $\mathcal{C}^k$ functions from I into $\mathbb{X}^1$ with bounded derivatives up to order k, equipped with the usual supremum norm (see section 2.2 for the definition of function spaces $\mathbb{X}^k$).

Lemma 3.1. *Consider any solution $a \in L^\infty(I, \ell_p)$ to equation (2.44), the associated leading order solution $\mathcal{Y}_0$ of (2.33) defined by (2.36), and its corrector $\mathcal{Y}_1$ defined by (2.40). There exist $M_0, M_1 > 0$ such that*

$$(3.2) \qquad \|\mathcal{Y}_0\|_{\mathcal{C}_b^1(I, \mathbb{X}^1)} \leq M_0 \, (\|a\|_{L^\infty(I,\ell_p)} + \|a\|_{L^\infty(I,\ell_p)}^\alpha),$$

$$(3.3) \qquad \|\mathcal{Y}_1\|_{\mathcal{C}_b^1(I, \mathbb{X}^1)} \leq M_1 \, (\|a\|_{L^\infty(I,\ell_p)}^\alpha + \|a\|_{L^\infty(I,\ell_p)}^{2\alpha-1}).$$

Proof. We immediately deduce from Lemma 2.7 that $\mathcal{Y}_0 \in \mathcal{C}_b^2(I, \mathbb{X}^1)$ and

$$(3.4) \qquad \|\mathcal{Y}_0\|_{L^\infty(I, \mathbb{X}^k)} \leq (1+k)\sqrt{2} \, \|a\|_{L^\infty(I,\ell_p)}, \quad k = 0, 1.$$

In what follows we estimate $G_\alpha(\mathcal{Y}_0)$, in order to estimate $\partial_\tau \mathcal{Y}_0$ from equation (2.38) and $\mathcal{Y}_1$ from equation (2.40). Since $G_\alpha \in \mathcal{C}^1(\ell_p^2, \ell_p^2)$ we have also $G_\alpha \in \mathcal{C}^1(\mathbb{X}^0, \mathbb{X}^0)$ by the omega-lemma (see [AMR88], lemma 2.4.18). Moreover, standard estimates yield for all $X \in \ell_p^2$

$$(3.5) \qquad \|G_\alpha(X)\|_p \leq M_3 \|X\|_p^\alpha, \quad \|DG_\alpha(X)\|_{\mathcal{L}(\ell_p^2)} \leq M_4 \|X\|_p^{\alpha-1},$$

which implies for all $X \in \mathbb{X}^0$

$$\|G_\alpha(X)\|_{\mathbb{X}^0} \leq M_3 \|X\|_{\mathbb{X}^0}^\alpha, \quad \|DG_\alpha(X)\|_{\mathcal{L}(\mathbb{X}^0)} \leq M_4 \|X\|_{\mathbb{X}^0}^{\alpha-1}.$$

We have then

$$\|G_\alpha(\mathcal{Y}_0)\|_{L^\infty(I,\mathbb{X}^0)} \leq M_3 \|\mathcal{Y}_0\|_{L^\infty(I,\mathbb{X}^0)}^\alpha,$$

$$\|\partial_\tau \mathcal{Y}_0\|_{L^\infty(I,\mathbb{X}^0)} \leq \|P\|_{\mathcal{L}(\mathbb{X}^0)} \|G_\alpha(\mathcal{Y}_0)\|_{L^\infty(I,\mathbb{X}^0)} \leq M_3 \|\mathcal{Y}_0\|_{L^\infty(I,\mathbb{X}^0)}^\alpha,$$

$$\|\partial_\tau G_\alpha(\mathcal{Y}_0)\|_{L^\infty(I,\mathbb{X}^0)} \leq M_4 \|\mathcal{Y}_0\|_{L^\infty(I,\mathbb{X}^0)}^{\alpha-1} \|\partial_\tau \mathcal{Y}_0\|_{L^\infty(I,\mathbb{X}^0)} \leq M_3 M_4 \|\mathcal{Y}_0\|_{L^\infty(I,\mathbb{X}^0)}^{2\alpha-1}$$

(the estimate of $\partial_\tau \mathcal{Y}_0$ follows from equation (2.38) and $\|P\|_{\mathcal{L}(\mathbb{X}^0)} = 1$). From these estimates and (3.4), there exists $M_5 > 0$ such that

$$(3.6) \qquad \|\partial_\tau \mathcal{Y}_0\|_{L^\infty(I,\mathbb{X}^1)} \leq M_5 \|a\|_{L^\infty(I,\ell_p)}^\alpha,$$

$$(3.7) \qquad \|G_\alpha(\mathcal{Y}_0)\|_{L^\infty(I,\mathbb{X}^0)} \leq M_5 \|a\|_{L^\infty(I,\ell_p)}^\alpha,$$

$$(3.8) \qquad \|\partial_\tau G_\alpha(\mathcal{Y}_0)\|_{L^\infty(I,\mathbb{X}^0)} \leq M_5 \|a\|_{L^\infty(I,\ell_p)}^{2\alpha-1}.$$

Consequently, estimate (3.2) is established thanks to (3.4) and (3.6). Furthermore, using the fact that $\mathcal{K} \in \mathcal{L}(\mathbb{X}^0, \mathbb{X}^1)$, we obtain $\mathcal{Y}_1 = \mathcal{K}G_\alpha(\mathcal{Y}_0) \in \mathcal{C}_b^1(I, \mathbb{X}^1)$ and

$$\|\mathcal{Y}_1\|_{\mathcal{C}_b^1(I,\mathbb{X}^1)} \leq \|\mathcal{K}\|_{\mathcal{L}(\mathbb{X}^0,\mathbb{X}^1)} M_5 (\|a\|_{L^\infty(I,\ell_p)}^\alpha + \|a\|_{L^\infty(I,\ell_p)}^{2\alpha-1}),$$

which establishes estimate (3.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Now we prove the main result of this section. The subsequent estimates will involve c.n.d.f. of various norms which we will denote by C_k.

Lemma 3.2. *There exist a c.n.i.f. ε_1 and a c.n.d.f. C_E such that for all $a \in L^\infty(I, \ell_p)$ solution to equation (2.44) and $\varepsilon \leq \varepsilon_1(\|a\|_{L^\infty(I,\ell_p)})$, the residual E^ε defined by (3.1) satisfies*

$$(3.9) \qquad \sup_{\varepsilon^{\alpha-1} t \in I} \|E^\varepsilon(t)\|_p \leq C_E(\|a\|_{L^\infty(I,\ell_p)})\varepsilon^{\alpha+\eta},$$

where $\eta = \min(\alpha - 1, \beta, \gamma)$.

Proof. Let us compute E^ε. Using identities (2.41), (2.35), (2.37) and (2.31), one obtains after some elementary computations

$$(3.10) \qquad \begin{aligned} E^\varepsilon(t) &= \varepsilon^\alpha[G_\alpha(Y_0(\varepsilon^{\alpha-1}t, t)) - G_\alpha(Y^\varepsilon(\varepsilon^{\alpha-1}t, t))] \\ &\quad + \varepsilon^{2\alpha-1}\partial_\tau Y_1(\varepsilon^{\alpha-1}t, t) - \varepsilon^\alpha R_\varepsilon(Y^\varepsilon(\varepsilon^{\alpha-1}t, t)). \end{aligned}$$

Let us estimate each term of (3.10) separately. From lemma 3.1 we already know a c.n.d.f. C_1 such that

$$(3.11) \qquad \sup_{\varepsilon^{\alpha-1}t\in I} \|\partial_\tau Y_1(\varepsilon^{\alpha-1}t,t)\|_p \leq C_1(\|a\|_{L^\infty(I,\ell_p)}).$$

Moreover, by estimate (2.32) and lemma 3.1, there exists a c.n.i.f. ε_1 and a c.n.d.f. C_2 such that for $\varepsilon \leq \varepsilon_1(\|a\|_{L^\infty(I,\ell_p)})$

$$(3.12) \qquad \sup_{\varepsilon^{\alpha-1}t\in I} \|R_\varepsilon(Y^\varepsilon(\varepsilon^{\alpha-1}t,t))\|_p \leq \varepsilon^{\min(\beta,\gamma)}C_2(\|a\|_{L^\infty(I,\ell_p)}).$$

Let us impose $\varepsilon_1 \leq 1$ without loss of generality. The first term at the right side of (3.10) can be estimated as follows for $\varepsilon^{\alpha-1}t \in I$, using estimate (3.5) and lemma 3.1 :

$$\|G_\alpha(Y_0(\varepsilon^{\alpha-1}t,t)) - G_\alpha(Y^\varepsilon(\varepsilon^{\alpha-1}t,t))\|_p$$
$$= \varepsilon^{\alpha-1}\Big\| \int_0^1 DG_\alpha(Y_0(\varepsilon^{\alpha-1}t,t) + \theta\varepsilon^{\alpha-1}Y_1(\varepsilon^{\alpha-1}t,t))\,\mathrm{d}\theta \cdot Y_1(\varepsilon^{\alpha-1}t,t)\Big\|_p$$
$$\leq \varepsilon^{\alpha-1} \int_0^1 \|DG_\alpha(Y_0(\varepsilon^{\alpha-1}t,t) + \theta\varepsilon^{\alpha-1}Y_1(\varepsilon^{\alpha-1}t,t))\|_{\mathcal{L}(\ell_p^2)}\|Y_1(\varepsilon^{\alpha-1}t,t)\|_p$$
$$\leq \varepsilon^{\alpha-1}C_3(\|Y_0(\varepsilon^{\alpha-1}t,t)\|_p + \|Y_1(\varepsilon^{\alpha-1}t,t)\|_p)\,\|Y_1(\varepsilon^{\alpha-1}t,t)\|_p$$
$$\leq \varepsilon^{\alpha-1}C_4(\|a\|_{L^\infty(I,\ell_p)}).$$

Combining this estimate with (3.11) and (3.12) yields the final estimate (3.9). $\qquad\square$

Example 3.3. In the case $\alpha = 3/2$ (as for the classical Hertz force), if W and ϕ are $\mathcal{C}^3$ at both sides of the origin, then $\beta,\gamma \geq 1/2$ in (2.2) and (2.3), and we have by lemma 3.2

$$\sup_{\varepsilon^{1/2}t\in I} \|E^\varepsilon(t)\|_p \leq C_\mathrm{E}(\|a\|_{L^\infty(I,\ell_p)})\varepsilon^2.$$

3.2. Proof of theorem 2.15. To prove theorem 2.15, we first estimate the error between the approximate solution $X_\mathrm{app}^\varepsilon$ constructed from a given $a(0) = a^0 \in \ell_p(\mathbb{Z},\mathbb{C})$ (equation (2.41)) and the exact solution X to equation (2.5) for $X(0) \approx X_\mathrm{app}^\varepsilon(0)$, in the case when $\varepsilon \approx 0$ and on $\mathcal{O}(\varepsilon^{1-\alpha})$ time scales. This result will follow in a rather standard way from Gronwall estimates. In a second step, we check on these time scales the validity of the leading-order approximate solution X_a^ε (equation (2.42)).

By lemma 2.7, for all initial condition $a^0 \in \ell_p(\mathbb{Z},\mathbb{C})$ equation (2.44) admits a unique maximal solution a defined for $\tau \in (T_\mathrm{min}, T_\mathrm{max})$. Let us fix $T \in (0, T_\mathrm{max})$ and restrict a to $[0,T]$, so that $a \in L^\infty([0,T],\ell_p)$. From lemma 3.1 it follows that $X_\mathrm{app}^\varepsilon \in L^\infty([0,t_\mathrm{m}(\varepsilon)],\ell_p^2)$ with $t_\mathrm{m}(\varepsilon) = T/\varepsilon^{\alpha-1}$, and we have $E^\varepsilon \in L^\infty([0,t_\mathrm{m}(\varepsilon)],\ell_p^2)$ by lemma 3.2.

Let $Z = X - X_\mathrm{app}^\varepsilon$ and $Z_0 \equiv X(0) - X_\mathrm{app}^\varepsilon(0)$. Then Z is solution to

$$\dot{Z} - JZ = G(X_\mathrm{app}^\varepsilon + Z) - G(X_\mathrm{app}^\varepsilon) - E^\varepsilon,$$
$$Z(0) = Z_0,$$

or equivalently

$$(3.13) \qquad Z(t) = e^{Jt}Z_0 + \int_0^t e^{J(t-s)}[G(X_\mathrm{app}^\varepsilon + Z) - G(X_\mathrm{app}^\varepsilon) - E^\varepsilon](s)\,\mathrm{d}s.$$

By Cauchy–Lipschitz theorem, the solution $Z \in \mathcal{C}^1([0, t_{\max}(\varepsilon)], \ell_p^2)$ to this equation is defined up to some maximal existence time $t_{\max}(\varepsilon) \leq t_{\mathrm{m}}(\varepsilon)$, depending *a priori* on Z_0.

Remark 3.4. In fact we prove below that $t_{\max}(\varepsilon) = t_{\mathrm{m}}(\varepsilon)$ when ε and $\|Z_0\|_p$ are small enough. This will provide a solution X to (2.5) defined at least for $t \in [0, t_{\mathrm{m}}(\varepsilon)]$.

We have already estimated the residual E^ε in the previous section. We now need to estimate the difference $G(X_{\mathrm{app}}^\varepsilon + Z) - G(X_{\mathrm{app}}^\varepsilon)$. For this purpose we assume $\|Z_0\|_p < \varepsilon$ and define

$$t_\varepsilon(Z_0) = \sup\{t \in [0, t_{\max}], \ \|Z(t)\|_p \leq \varepsilon\}.$$

Remark 3.5. Since $\|Z\|_p \in \mathcal{C}^0([0, t_{\max}])$, we have either $t_\varepsilon = t_{\max}$ or $t_\varepsilon < t_{\max}$ and $\|Z(t_\varepsilon)\|_p = \varepsilon$.

Lemma 3.6. *There exists a c.n.i.f. ε_2 and a c.n.d.f. C_{L} such that for all solution $a \in L^\infty([0, T], \ell_p)$ to equation (2.44), $\varepsilon \leq \varepsilon_2(\|a\|_{L^\infty([0,T],\ell_p)})$, $Z_0 \in \ell_p^2$ with $\|Z_0\|_p < \varepsilon$ and $t \in [0, t_\varepsilon(Z_0)]$,*

$$(3.14) \qquad \|G(X_{\mathrm{app}}^\varepsilon(t) + Z(t)) - G(X_{\mathrm{app}}^\varepsilon(t))\|_p \leq \varepsilon^{\alpha-1} C_{\mathrm{L}}(\|a\|_{L^\infty([0,T],\ell_p)}) \|Z(t)\|_p.$$

Proof. We first estimate
(3.15)
$$\|G(X_{\mathrm{app}}^\varepsilon(t) + Z(t)) - G(X_{\mathrm{app}}^\varepsilon(t))\|_p \leq \int_0^1 \|DG(X_{\mathrm{app}}^\varepsilon(t) + \theta Z(t))\|_{\mathcal{L}(\ell_p^2)} \mathrm{d}\theta \, \|Z(t)\|_p.$$

By estimate (2.7), there exists $\mu, C > 0$ such that for all $U \in \ell_p^2$ with $\|U\|_p \leq \mu$, we have

$$(3.16) \qquad \|DG(U)\|_{\mathcal{L}(\ell_p^2)} \leq C \|U\|_p^{\alpha-1}.$$

From definition (2.41) and lemma 3.1, assuming $\varepsilon_2 \leq 1$, there exists a c.n.d.f. C_5 such that

$$\sup_{t \in [0, t_{\mathrm{m}}(\varepsilon)]} \|X_{\mathrm{app}}^\varepsilon(t)\|_p \leq \varepsilon \, C_5(\|a\|_{L^\infty([0,T],\ell_p)}) \quad \text{for all } \varepsilon \in (0, \varepsilon_2],$$

hence for all $\theta \in [0, 1]$ and $t \in [0, t_\varepsilon(Z_0)]$

$$(3.17) \qquad \|X_{\mathrm{app}}^\varepsilon(t) + \theta Z(t)\|_p \leq \varepsilon(1 + C_5(\|a\|_{L^\infty([0,T],\ell_p)})).$$

Consequently, there exists a c.n.i.f. $\varepsilon_2 \leq 1$ such that for all $\varepsilon \leq \varepsilon_2(\|a\|_{L^\infty([0,T],\ell_p)})$, $\theta \in [0, 1]$ and $t \in [0, t_\varepsilon(Z_0)]$ we have $\|X_{\mathrm{app}}^\varepsilon(t) + \theta Z(t)\|_p \leq \mu$. Then one obtains estimate (3.14) using bounds (3.16) and (3.17) in conjunction with estimate (3.15). $\qquad\square$

Now let us apply lemmas 3.2 and 3.6 to the Duhamel formulation (3.13) for $Z(t)$. This yields for all $\varepsilon \leq \varepsilon_2(\|a\|_{L^\infty([0,T],\ell_p)})$, $Z_0 \in \ell_p^2$ with $\|Z_0\|_p < \varepsilon$ and for all $t \in [0, t_\varepsilon]$

$$\|Z(t)\|_p \leq b_\varepsilon + C_{\mathrm{L}} \varepsilon^{\alpha-1} \int_0^t \|Z(s)\|_p \, \mathrm{d}s,$$

with $b_\varepsilon = \|Z_0\|_p + C_{\mathrm{E}} T \varepsilon^{1+\eta}$. Then one obtains by Gronwall lemma

$$(3.18) \qquad \|Z(t)\|_p \leq b_\varepsilon \exp(C_{\mathrm{L}} \varepsilon^{\alpha-1} t_\varepsilon) \leq b_\varepsilon \exp(C_{\mathrm{L}} T).$$

We now fix a c.n.d.f. $C_0 > 0$ and make the following stronger assumption on the initial distance between exact and approximate solutions.

Assumption 3.7. $Z_0 = X(0) - X_{\mathrm{app}}^\varepsilon(0)$ satisfies

$$\|Z_0\|_p \leq C_0(\|a\|_{L^\infty([0,T],\ell_p)}) \, T \, \varepsilon^{1+\eta}.$$

Assumption 3.7 implies $\|Z_0\|_p < \varepsilon$ as soon as $\varepsilon < \varepsilon_3 = (C_0 \, T)^{-1/\eta}$, and then estimate (3.18) applies for $\varepsilon < \min(\varepsilon_2, \varepsilon_3)$. This yields

$$(3.19) \qquad \|Z(t)\|_p \leq C_{\mathrm{R}}(\|a\|_{L^\infty([0,T],\ell_p)}) \, \varepsilon^{1+\eta}, \quad t \in [0, t_\varepsilon],$$

where $C_{\mathrm{R}} = (C_0 + C_{\mathrm{E}}) \, T \, \exp(C_{\mathrm{L}}T)$ is a c.n.d.f.

Consequently, for $\varepsilon < \varepsilon_T = \min(\varepsilon_2, C_{\mathrm{R}}^{-1/\eta})$ we have $\|Z(t)\|_p < \varepsilon$ for all $t \in [0, t_\varepsilon]$, which implies $t_\varepsilon = t_{\max}$ (see remark 3.5). Now estimate (3.19) allows to bound $\|Z(t)\|_p$ for $t \in [0, t_{\max}]$. Consequently, for all $\varepsilon < \varepsilon_T(\|a\|_{L^\infty([0,T],\ell_p)})$ and Z_0 satisfying assumption 3.7 we have $t_{\max}(\varepsilon) = t_{\mathrm{m}}(\varepsilon)$ and

$$(3.20) \qquad \|X(t) - X_{\mathrm{app}}^\varepsilon(t)\|_p \leq C_{\mathrm{R}}(\|a\|_{L^\infty([0,T],\ell_p)}) \, \varepsilon^{1+\eta}, \quad t \in [0, T/\varepsilon^{\alpha-1}].$$

With the error estimate (3.20) at hand, one can recover an estimate of the same type for the leading order approximate solution $X_a^\varepsilon(t) = \varepsilon Y_0(\varepsilon^{\alpha-1}t, t)$. Indeed

$$(3.21) \qquad X(t) - X_a^\varepsilon(t) = X(t) - X_{\mathrm{app}}^\varepsilon(t) + \varepsilon^\alpha Y_1(\varepsilon^{\alpha-1}t, t).$$

From lemma 3.1, we already know a c.n.d.f. C_1 such that for all $t \in [0, T/\varepsilon^{\alpha-1}]$,

$$(3.22) \qquad \|Y_1(\varepsilon^{\alpha-1}t, t)\|_p \leq C_1(\|a\|_{L^\infty([0,T],\ell_p)}).$$

Now let us set $C_0 = (C_{\mathrm{i}} + C_1)/T$ in assumption 3.7, where $C_{\mathrm{i}} > 0$ is a fixed constant, and make the assumption $\|X^0 - X_a^\varepsilon(0)\|_p \leq C_{\mathrm{i}}\varepsilon^{1+\eta}$ of theorem 2.15. Recalling that $\varepsilon \leq 1$, $\eta + 1 \leq \alpha$, and using (3.21)–(3.22), one can check that assumption 3.7 is satisfied and thus estimate (3.20) holds. Then using (3.21)–(3.22) again, there exists a c.n.d.f. $C_T = C_{\mathrm{R}} + C_1$ such that for all $t \in [0, T/\varepsilon^{\alpha-1}]$,

$$\|X(t) - X_a^\varepsilon(t)\|_p \leq C_T(\|a\|_{L^\infty([0,T],\ell_p)}) \, \varepsilon^{1+\eta}.$$

This completes the proof of theorem 2.15.

3.3. Proof of theorem 2.20. Theorem 2.20 consists of a different kind of error estimate, where the multiple-scale approximation is controlled on $\mathcal{O}(|\ln \varepsilon| \, \varepsilon^{1-\alpha})$ time scales going beyond $\varepsilon^{1-\alpha}$, at the expense of lowering the precision of (2.56). To obtain this result, we adapt the proof of theorem 2.15 by allowing T to grow logarithmically in ε. Theorem 2.20 is valid when the Ansatz $X_a^\varepsilon(t) \in \ell_p^2$ is bounded for $t \in \mathbb{R}^+$. In that case, all estimates of sections 3.1 and 3.2 involving constants depending only on $\|a\|_{L^\infty([0,T],\ell_p)}$ can be made uniform w.r.t. T, which allows one to set $T = \mathcal{O}(|\ln \varepsilon|)$ when $\varepsilon \to 0$.

In what follows we use the notations and definitions introduced in section 3.2. Let us consider a solution $a \in L^\infty([0, +\infty), \ell_p)$ of the amplitude equation (2.43), a fixed constant $\mu \in (0, \eta)$ and set

$$T = \frac{\mu}{\tilde{C}_{\mathrm{L}}} \, |\ln \varepsilon|,$$

where $\tilde{C}_{\mathrm{L}} = C_{\mathrm{L}}(\|a\|_{L^\infty([0,+\infty),\ell_p)})$ and $\varepsilon \leq 1$. The initial error Z_0 is set to satisfy assumption 3.7 with $C_0 = M$, M being a fixed constant that will be subsequently

determined. It follows that $\|Z_0\|_p = \mathcal{O}(|\ln\varepsilon|\,\varepsilon^{1+\eta}) < \varepsilon$ provided ε is small enough. Assuming in addition $\varepsilon \le \tilde{\varepsilon}_2 = \varepsilon_2(\|a\|_{L^\infty([0,+\infty),\ell_p)})$, estimate (3.19) ensures that

$$(3.23) \qquad \|Z(t)\|_p \le (M + \tilde{C}_{\mathrm{E}})\,T\,\exp(\tilde{C}_{\mathrm{L}}T)\,\varepsilon^{1+\eta}, \quad t \in [0, t_\varepsilon],$$

where $\tilde{C}_{\mathrm{E}} = C_{\mathrm{E}}(\|a\|_{L^\infty([0,+\infty),\ell_p)})$. Our choice of T yields exactly $\exp(\tilde{C}_{\mathrm{L}}\,T) = \varepsilon^{-\mu}$, hence estimate (3.23) becomes

$$(3.24) \qquad \|Z(t)\|_p \le \frac{M + \tilde{C}_{\mathrm{E}}}{\tilde{C}_{\mathrm{L}}}\,\mu\,|\ln\varepsilon|\,\varepsilon^{1+\eta-\mu}, \quad t \in [0, t_\varepsilon].$$

Consequently, for ε small enough we have $\|Z(t)\|_p < \varepsilon$ for all $t \in [0, t_\varepsilon]$, which implies $t_\varepsilon = t_{\max} = T\,\varepsilon^{1-\alpha}$ as shown in section 3.2.

Now, as previously observed in section 3.2, the error bound (3.24) yields an estimate of the same type for the leading order approximate solution $X_a^\varepsilon(t)$, thanks to the estimate

$$(3.25) \qquad \|Y_1(\varepsilon^{\alpha-1}t, t)\|_p \le \tilde{C}_1, \quad t \in [0, T/\varepsilon^{\alpha-1}],$$

with $\tilde{C}_1 = C_1(\|a\|_{L^\infty([0,\infty),\ell_p)})$. Indeed, let us further assume $\varepsilon \le e^{-1}$ and make the assumption $\|X^0 - X_a^\varepsilon(0)\|_p \le C_{\mathrm{i}}\,|\ln\varepsilon|\,\varepsilon^{1+\eta}$ of theorem 2.20 for some fixed constant C_{i}. Using identity (3.21) and the bounds given above, one obtains

$$\|X^0 - X_{\mathrm{app}}^\varepsilon(0)\|_p \le (C_{\mathrm{i}} + \tilde{C}_1)\,|\ln\varepsilon|\,\varepsilon^{1+\eta}$$

(we recall that $\eta \le \alpha - 1$). Consequently, assumption 3.7 is satisfied with the choice $C_0 = M = (C_{\mathrm{i}} + \tilde{C}_1)\tilde{C}_{\mathrm{L}}/\mu$ and estimate (3.24) holds true for all $t \in [0, T\varepsilon^{1-\alpha}]$. Then using (3.21)-(3.25) and the definition of T, we get for all $t \in [0, \mu\,\tilde{C}_{\mathrm{L}}^{-1}\,|\ln\varepsilon|\,\varepsilon^{1-\alpha}]$

$$\|X(t) - X_a^\varepsilon(t)\|_p \le C_{\mathrm{l}}\,|\ln\varepsilon|\,\varepsilon^{1+\eta-\mu},$$

with $C_{\mathrm{l}} = C_{\mathrm{i}} + 2\tilde{C}_1 + \eta\,\tilde{C}_{\mathrm{E}}/\tilde{C}_{\mathrm{L}}$, which proves estimate (2.58) for $\nu = \tilde{C}_{\mathrm{L}}^{-1}$. This ends the proof of theorem 2.20.

4. Conclusion. We have shown that small amplitude oscillations in Newton's cradle are described by the DpS equation (1.2) over long times. From this result, we have estimated on long time scales the maximal decay of small amplitude localized solutions and proved the existence of stable long-lived breather states in Newton's cradle.

The justification of the DpS equation and the associated estimates of maximal decay extend straightforwardly to generalizations of (2.4) and (1.2) to arbitrary space dimensions (i.e. for $n \in \mathbb{Z}^d$ and $d \ge 1$) when $x_n(t)$ defines a scalar field. However, generalizing our construction of long-lived breather states would require an existence theorem for discrete breather solutions of the d-dimensional DpS equation, which is not yet available for $d \ge 2$. Other possible extensions of this work concern the generalization and justification of the DpS equation when small spatial inhomogeneities are present in the original lattice (2.4), as well as the addition of dissipative terms in (2.4) and (1.2). Considering these effects is particularly important from a physical point of view when system (2.4) describes a granular chain [JKC13].

Other open problems concern the qualitative analysis of the DpS equation. In particular, excitations generated from a localized disturbance and reminiscent of traveling breather solutions have been numerically studied in [JKC13, SHVM12], both for

the DpS equation and Newton's cradle. The existence of exact traveling breather solutions of (1.2) is an open problem, and would imply (in the case of small amplitude waves) the existence of similar excitations in Newton's cradle on long time scales. More generally, understanding in system (1.2) the complex mechanisms of fully nonlinear energy propagation from a localized disturbance is a challenging open problem [JKC13]. This would allow in particular to analyze the propagation of nonlinear acoustic waves after an impact in granular chains with local potentials, thanks to the connection we have established between (1.2) and (1.1).

Appendix. Simplified form of the amplitude equation.

This section provides an explicit computation of the nonlinear term of the amplitude equation (2.38). Given the form (2.36) of $\mathcal{Y}_0$ and recalling that $P(X) = \zeta(X)e^{it}e_1 +$ c.c. (see lemma 2.4), the amplitude $a(\tau) \in \ell_p(\mathbb{Z}, \mathbb{C})$ satisfies the differential equation

$$\partial_\tau a = \zeta(G_\alpha(\mathcal{Y}_0)),$$

or more explicitly

$$i\partial_\tau a = \delta^+ f(\delta^- a),$$

where

$$\forall\, a \in \ell_p(\mathbb{Z}, \mathbb{C}),\ \ (f(a))_n = \tilde{f}(a_n),$$
$$\forall\, a \in \mathbb{C},\ \ \tilde{f}(a) = \frac{1}{\sqrt{2}}\frac{1}{2\pi}\int_0^{2\pi} e^{-it}V_\alpha'\left(\frac{a\,e^{it}}{\sqrt{2}} + \text{c.c.}\right) \mathrm{d}t.$$

Below we compute the map $\tilde{f}$ explicitly.

Setting $a = r\,e^{i\theta}$ and using the change of variable $s = t+\theta$ in the integral defining $\tilde{f}$, one obtains

$$\tilde{f}(a) = \frac{e^{i\theta}}{2^{3/2}\,\pi}\int_{S^1} V_\alpha'(\sqrt{2}\,r\cos s)\,e^{-is}\,\mathrm{d}s,$$

where one can fix $S^1 = (-\pi, \pi)$. Given the form (2.1) of the potential V_α, one obtains after elementary computations,

$$\tilde{f}(a) = 2^{\frac{\alpha-3}{2}}\,(k_- + k_+)\,c_\alpha\,a\,|a|^{\alpha-1},$$

where

$$c_\alpha = \frac{2}{\pi}\int_0^{\pi/2}(\cos t)^{\alpha+1}\,\mathrm{d}t$$

is a Wallis integral with fractional power $\alpha + 1$. Expressing c_α in terms of Euler's Gamma function leads to

$$c_\alpha = \frac{1}{\pi}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{\alpha}{2}+1)}{\Gamma(\frac{\alpha+1}{2}+1)}$$

(see [AS70], formula 6.2.1 and 6.2.2, p. 258). Since $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(a+1) = a\,\Gamma(a)$, we obtain finally

$$c_\alpha = \frac{\alpha\,\Gamma(\frac{\alpha}{2})}{\sqrt{\pi}(\alpha+1)\Gamma(\frac{\alpha+1}{2})}.$$

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